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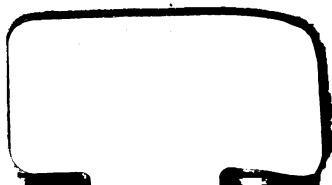
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AN
ANALYTICAL SYSTEM
OF
CONIC SECTIONS:

DESIGNED
FOR THE USE OF STUDENTS.

BY THE
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PREFACE

TO THE FIFTH EDITION.

THE chief design of the following pages is to facilitate the Study of Conic Sections, upon principles accommodated to the present state of Mathematical Science. In a former Publication, the Author presented a slight sketch of the same subject, viewed in connection with Analytical Geometry. But experience has since taught him, that this method of treating Conic Sections, although sanctioned by the practice of distinguished Continental Writers, is too scientific, if he may be allowed the expression, for elementary instruction. It is deficient in the essential quality of Simplicity, and presupposes a greater degree of expertness in Algebraical Operations than persons, at the time of their entering upon this branch of Study, are commonly found to have attained.

In attempting to supply these defects it was thought advisable, in the first place, to assume a Definition as the Basis of the following Work, instead of founding it upon the Sections of the Cone, or upon the Discussion of the general Equation of the second degree. The clearness and

precision which characterize this mode of introducing the Subject will, it is hoped, justify the Author for having, in the adoption of it, departed in some degree from the rigour of Analysis.

He begins then, with defining a Conic Section to be *the locus of a point, whose distances from a fixed point and a straight line given in position, are to each other in a constant ratio*. This definition is shewn to comprehend three curves, the Parabola, Ellipse and Hyperbola. In treating of these, each of them is referred, for the sake of distinctness, to rectangular, oblique, and polar Co-ordinates: and in order that the relations which they bear to each other may be presented in the clearest light, the same method, both of arrangement and of demonstration, is observed in deducing their several properties.

The Ellipse and Hyperbola are so intimately connected, that the properties of the one may always be derived from the analogous properties of the other, by changing the sign of a single letter in the fundamental Equation. For this reason the Author may seem to have been unnecessarily diffuse, in having allotted to each curve a separate investigation. He is inclined to believe, however, that the method here pursued will be found the best adapted to Beginners; and for such his work is principally intended.

The subject of Curvature, to which a place is usually assigned in Treatises of Conic Sections, is in this omitted: because it belongs, in strict propriety, to the Differential Calculus.

The latter part of the Volume contains a discussion of the general Equation of the second degree, without which an Analytical Work on Conic Sections can scarcely be considered as complete. This discussion embraces two objects: first to establish the identity of the Conic Sections with the Loci of that Equation: and secondly, to shew that the latter admits of being so reduced, as to coincide with the general Equation to the Conic Sections in its simplest form.

As a knowledge of the first principles of Analytical Geometry is requisite, before entering upon the main subject of the following Work, the Author has prefixed an Introduction, in which the Equations to the straight line and circle are treated of at considerable length.

Since the publication of the earlier Editions of the Work, various alterations and improvements have been introduced; but it is unnecessary to specify them, as a copious Analysis of the Volume is exhibited in the Table of Contents.

Great pains have been taken in rendering the present Edition typographically correct; and it is hoped that no errors of any consequence have escaped detection.

WATH RECTORY, NEAR RIPON,
October 16th, 1843.

ERRATA.

Page 31, line 3 from the top, *for as read as'.*

..... 32, ... 5 (21) (30).

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AN
ANALYTICAL SYSTEM
OF
CONIC SECTIONS.

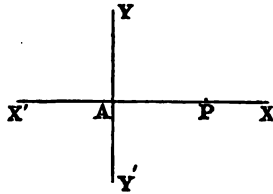
INTRODUCTION.

CHAPTER I.

ON THE POSITION OF A POINT.

1. *THE position of a point, situated on a straight line, is determined by referring it to any other point, assumed at pleasure, in the same line.*

Let XX' be a straight line of indefinite length, P any



point in it, A any other point assumed at pleasure in XX' ; then the position of P is determined by referring it to A .

If the variable distance AP be denoted by x , then the point P recedes from A towards X when x increases, it approaches from X towards A when x decreases, it coincides with A when $x = 0$, and it again recedes from A , in the opposite direction towards X' , when x is negative.

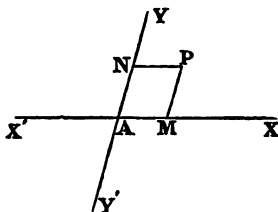
Hence if x be supposed *positive* when P lies to the *right* of the point A , it must be considered as *negative* when P lies to the *left* of A .

Again, let the line YAY' be perpendicular, or inclined at any known angle, to XX' , then it may, in like manner, be shewn that if x be supposed positive when P is situated on AY , *above* XX' , it must be considered as negative when P is situated on AY' , *below* XX' .

This conventional rule, by which *difference of sign* is made to indicate *contrariety of direction*, is applicable in all cases where distance is to be estimated from a fixed point, along a straight line given in position.

2. *The position of a point, situated on a plane, is determined by referring it to two straight lines, which intersect each other at a given angle, in the same plane.*

Let P be a point on a plane, XX' and YY' two straight



lines of indefinite length, and intersecting each other in a known angle A . From P draw PM , PN parallel respectively to AY and AX . Then the position of P will be determined by means of the lines PM , PN , which measure its distances from AX , AY .

The lines AM , MP , are called the *co-ordinates* of the point P , and A is said to be the *origin*, and AX , AY the *axes*, of the co-ordinates.

When the co-ordinates AM , MP , are unknown, AM is denoted by x , and MP by y ; when they are supposed to be known, they are usually denoted by the accented letters x' , y' , or x'' , y'' .

Of the two co-ordinates AM , MP , the former is, for the sake of distinction, termed the *abscissa*, and the latter the *ordinate*. The line AX , along which the abscissas are measured, is called *the axis of x* , and the line AY , in the direction of which the ordinates are measured, *the axis of y* .

The axes AX , AY are generally drawn at right angles to each other, and are then denominated *rectangular axes*; in all other cases they are called *oblique axes*.

3. When the co-ordinates of a point are given, the signs with which they are affected will serve to determine in which of the four angles, formed by the intersection of the axes at the origin A , that point is situated.

Let x' , y' be the co-ordinates of a point P ; Fig. p. 2. then,

- (1) When P is any where within the angle YAX , x' and y' are both positive.

When it is situated on the axis AX , x' is positive, and $y' = 0$, and when it is situated on the axis AY , $x' = 0$, and y' is positive.

- (2) When P is any where within the angle YAX' , x' is negative, and y' positive, and when it is situated on the axis AX' , x' is negative, and $y' = 0$.
- (3) When P is any where within the angle $Y'AX$, x' and y' are both negative, and when it is situated on the axis AY' , $x' = 0$, and y' is negative.
- (4) When P is any where within the angle $Y'AX'$, x' is positive, and y' negative.

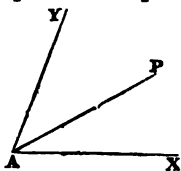
4. If a line, straight or curved, be traced upon a plane, and from the definition, or some known property, of the line, a relation be deduced between the co-ordinates of any one of its points, then the indeterminate equation which expresses this relation, supposing it to be the same for every point, is called THE EQUATION TO THE LINE.

Conversely, an indeterminate equation between two variables being given, a series of points may be found which correspond to the various solutions of that equation: the assemblage of points so determined is called the LOCUS of the proposed equation.

The meaning of these two paragraphs will be fully illustrated, when we come to treat of the straight line and circle.

5. The position of a point upon a plane may be determined not only by means of its co-ordinates, but also by means of its distance from a fixed point, and of the angle which that distance makes with a straight line given in position.

Thus, let P be any point on a plane, A a fixed point, AX

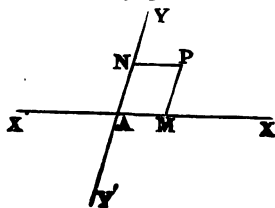


a straight line given in position, and let the points A , P be joined. Then the position of P will manifestly be given, when the distance AP and the angle PAX which it makes with AX are known.

The fixed point A is called the *pole*, and the variable distance AP , the *radius vector*; AP is usually denoted by r , and the angle PAX by ω . The quantities r and ω are termed *polar co-ordinates*, and the equation expressing the relation between them at any point of a curve is called the *polar equation to the curve*.

6. To find the equation to a point.

Let the co-ordinates of any point P be x' , y' .



Then its equation will be

$$x = x', \quad y = y',$$

when P is within the angle YAX , Art. 3;

$$\text{or } x = -x', \quad y = y',$$

when P is within YAX' ;

$$\text{or } x = -x', \quad y = -y',$$

when P is within $Y'AX'$;

$$\text{or } x = x', \quad y = -y',$$

when P is within $Y'AX$.

7. COR. 1. When P is situated on the axis of x , $y = 0$; therefore its equation is

$$x = \pm x', \quad y = 0;$$

and similarly, when P is on the axis of y , its equation is

$$x = 0, \quad y = \pm y'.$$

8. COR. 2. When the point P coincides with the origin A ,

$$x = 0, \quad \text{and } y = 0.$$

CHAPTER II.

ON THE STRAIGHT LINE.

SECTION I.

ON THE EQUATION TO THE STRAIGHT LINE IN GENERAL.

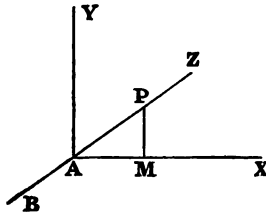
9. *To find the equation to a straight line.*

Let BZ be a straight line of indefinite length, whose equation it is required to find.

The inclination and position of the axes being arbitrary, we shall suppose that they are rectangular, and

1st. That their origin A is a point in the proposed line.

Through A draw AX , making a known angle with BZ ,



and let AY be at right angles to AX : assume any point P in BZ , and let fall the perpendicular PM on AX .

Let $AM = x$, $MP = y$.

Then $\frac{PM}{AM} = \tan ZAX$;

$$\therefore \frac{y}{x} = \tan ZAX;$$

but the angle ZAX being supposed known, we may denote its tangent by a ;

$$\therefore \frac{y}{x} = a.$$

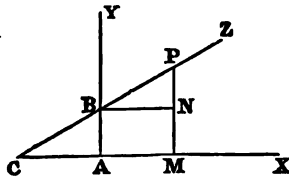
As the same relation may be proved to subsist between the co-ordinates of every other point in the line, the equation required is (4)

$$y = ax.$$

Again let us suppose

2dly. That the origin of the axes is without the proposed line, as at A .

Let BZ meet AX in C , and AY in B : and through B



draw BN parallel to AX meeting PM , the ordinate of P , in N .

As before, let $AM = x$, $MP = y$, $\tan ZCX = a$; also let $AB = b$.

$$\text{Then } \frac{PN}{NB} = \tan PBN$$

$$= \tan ZCX;$$

$$\therefore PN = NB \cdot \tan ZCX;$$

$$\text{but } PN = PM - MN$$

$$= y - b;$$

$$\therefore y - b = ax,$$

$$\text{or } y = ax + b,$$

which is the equation required.

It thence appears that the equation to a straight line is

$$y = ax,$$

$$\text{or } y = ax + b,$$

according as it passes, or does not pass, through the origin.

10. **COR.** If the axes be inclined at *any* angle to each other, then conceiving the last figure adapted to oblique co-ordinates,

$$\frac{PN}{NB} = \frac{PM}{MC} = \frac{\sin PCM}{\sin CPM}.$$

Now PCM , CPM being the angles which the straight line BZ makes with the axes of x and y respectively, the ratio of their sines is known, and may therefore be assumed $= a$; hence the equation to BZ retains the same form

$$y = ax + b,$$

in which *the coefficient (a) now represents the ratio of the sines of the angles which BZ makes with the axes of x and y.*

11. **Observation.** If ρ be taken to represent the indefinite straight line BZ , and if the sines of the angles which it makes with the axes of x and y respectively, be denoted by $\sin \rho, x$ and $\sin \rho, y$, the above equation will become

$$y = \frac{\sin \rho, x}{\sin \rho, y} x + b.$$

In complicated formulas this self-interpreting notation will be found extremely convenient.

If γ denote the inclination of the axes to each other, and θ the inclination of the straight line to AX , then the above equation becomes

$$y = \frac{\sin \theta}{\sin (\gamma - \theta)} x + b.$$

12. Conversely, the locus of the indeterminate equation of the first degree,

$$Ay + Bx + C = 0,$$

is a straight line.

For dividing by A , and transposing, we have

$$y = -\frac{B}{A}x - \frac{C}{A},$$

and denoting $-\frac{B}{A}$ by a , and $-\frac{C}{A}$ by b ,

$$y = ax + b,$$

which coincides with the equation to a straight line, just obtained.

13. In employing the equation to a straight line we always suppose the line to be of indefinite length.

When we have occasion to consider *two* straight lines, instead of representing them by the equations

$$y = ax + b,$$

$$Y = a'X + b',$$

in which two sets of letters, x , y , and X , Y , are used for denoting the variable co-ordinates, it is customary to represent them by the equations

$$y = ax + b,$$

$$y = a'x + b',$$

in which the same letters, x , y , are used for the co-ordinates in both equations. But it must be observed that x and y have not the same value in these two equations, except in the particular case where the lines intersect each other; for then, as is evident, the co-ordinates at the point of intersection will be the same for each line.

14. When it is required to construct the locus of any indeterminate equation of the first degree, it is sufficient to find two of its points, because two points only are necessary to fix the position of a straight line. The two points most readily found are those in which the proposed line cuts the axes of x and y , and which are determined by making x and y successively $= 0$ in the given equation: in the first case, the resulting value of y represents the distance of the origin from the point of intersection with AY ; in the second case, the resulting value of x represents the distance of the origin from the point of intersection with AX . The indefinite straight line joining these points of intersection is the line required.

As an illustration of this process, let it now be required to assign the position of the straight line which is the locus of any indeterminate equation of the first degree.

The proposed equation may occur under the four following forms :

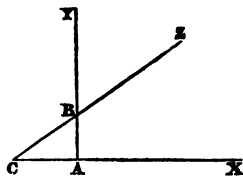
$$1. \quad y = ax + b,$$

$$2. \quad y = -ax + b,$$

$$3. \quad y = ax - b,$$

$$4. \quad y = -ax - b.$$

(1) To find the locus of $y = ax + b$.



Let $x = 0$; $\therefore y = b$.

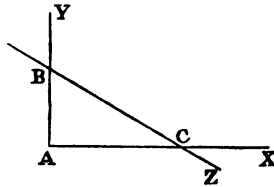
In AY take $AB = b$, which is therefore the value of y when $x = 0$.

Let $y = 0$; $\therefore x = -\frac{b}{a}$.

This being negative, in XA produced take $AC = \frac{b}{a}$ and it will be the value of x corresponding to $y = 0$.

Join B, C and produce BC both ways, then the indefinite line ZC is the locus required.

(2) To find the locus of $y = -ax + b$.



Let $x = 0$; $\therefore y = b$.

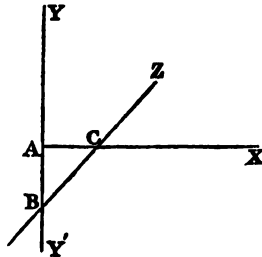
In AY take $AB = b$, which is the value of y when $x = 0$.

Let $y = 0$; $\therefore x = \frac{b}{a}$.

In AX take $AC = \frac{b}{a}$, which is the value of x corresponding to $y = 0$.

Join B, C , then the indefinite line BZ is the locus required.

(3) To find the locus of $y = ax - b$.



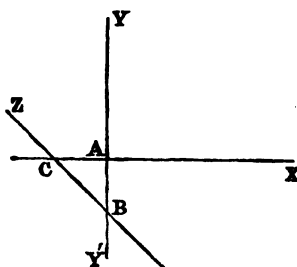
Let $x = 0$; $\therefore y = -b$.

In YA produced, take $AB = b$.

Let $y = 0$; $\therefore x = \frac{b}{a}$.

In AX take $AC = \frac{b}{a}$; join B, C , then BCZ is the locus required.

(4) To find the locus of $y = -ax - b$.



Let $x = 0$; $\therefore y = -b$.

In YA produced, take $AB = b$.

Let $y = 0$; $\therefore x = -\frac{b}{a}$.

In XA produced, take $AC = \frac{b}{a}$; join C, B , then BCZ is the line required.

Observation. This method fails when applied to the equation

$$y = ax, \text{ in which } b = 0.$$

But since, Art. 9, the straight line which is the locus of this equation must pass through the origin, we have only to determine the position of another point in the line, which is done by assigning any arbitrary value to x , and constructing the resulting value of y .

In the preceding problem we have supposed the axes to be rectangular, but the same method is applicable whatever be their inclination.

15. The equation to a straight line sometimes occurs under the form

$$\frac{x}{m} + \frac{y}{n} = 1;$$

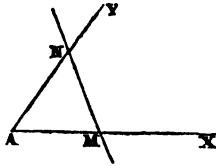
$$\text{or } my + nx = mn,$$

which is immediately deducible from the former equation.

Let $y = -ax + b$ be the equation to any indefinite straight line MN , then dividing each side by b , and transposing,

$$\frac{y}{b} + \frac{a}{b}x = 1.$$

But by case (2) p. 11, $\frac{b}{a} = AM$ and $b = AN$.



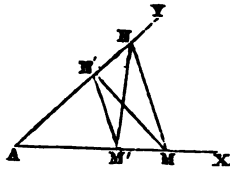
Let $AM = m$ and $AN = n$;

$$\therefore \frac{y}{n} + \frac{x}{m} = 1,$$

$$\text{or } my + nx = mn;$$

in which m represents the part of AX , and n the part of AY , cut off by the proposed line.

We have taken the case where m and n are both positive; when either or both of them are negative, the equation will undergo a corresponding modification.



16. COR. Let MN , $M'N'$ be any two straight lines, and let $AM = m$, $AN = n$,

$$AM' = m', \quad AN' = n'.$$

Then the equation to MN is $my + nx = mn$,

$$\dots\dots\dots M'N' \dots m'y + n'x = m'n',$$

$$\dots\dots\dots MN' \dots my + n'x = mn',$$

$$\dots\dots\dots M'N \dots m'y + nx = m'n.$$

SECTION II.

ON THE EQUATION TO THE STRAIGHT LINE SUBJECT TO
CERTAIN CONDITIONS.

17. *To find the equation to a straight line which passes through a given point.*

Let the co-ordinates of the given point be x', y' , and those of any other point whatever in the same line, x, y .

Then the equation to a straight line in general is

$$y = ax + b \dots \dots \dots (1),$$

and x', y' being the co-ordinates of a point in the line, they must satisfy this equation;

$$\therefore y' = ax' + b;$$

$$\therefore b = y' - ax'.$$

Substituting this value of b in (1), we have

$$y = ax + y' - ax',$$

$$\text{or } y - y' = a(x - x'),$$

which is the equation required.

The coefficient a is *indeterminate*, because an infinite number of straight lines may be drawn, all of which pass through the same point.

18. COR. 1. We have assumed x' and y' both positive; if both be negative, or if one be negative and the other positive, the equations will be, respectively,

$$y + y' = a(x + x'),$$

$$y - y' = a(x + x'),$$

$$y + y' = a(x - x').$$

19. COR. 2. If the given point be on the axis of x , then $y' = 0$, and the equation becomes

$$y = a(x - x');$$

and if the given point be on the axis of y , then $x' = 0$, and the equation becomes

$$y - y' = ax.$$

20. *To find the equation to a straight line which passes through two given points.*

Let the co-ordinates of the given points be x' , y' , and x'' , y'' , and those of any other point whatever in the same line, x , y .

Then, in general,

$$y = ax + b \dots \dots \dots (1),$$

and as the given co-ordinates must satisfy this equation, we also have

$$y' = ax' + b \dots \dots \dots (2),$$

$$\text{and } y'' = ax'' + b \dots \dots \dots (3).$$

Subtracting (2) from (1) we have $y - y' = a(x - x') \dots (4)$.

Now, since the position of a straight line which passes through two given points is fixed, the indeterminate quantity a in equation (4) must be expressed in terms of the

given co-ordinates. To do this, subtract (2) from (3), and we shall have

$$y'' - y' = a(x'' - x');$$

$$\therefore a = \frac{y'' - y'}{x'' - x'},$$

and substituting this in (4),

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x'),$$

which is the equation required.

21. COR. 1. If either of the points, for example the second, be on the axis of x , then $y'' = 0$, and the equation becomes

$$y - y' = -\frac{y'}{x'' - x'}(x - x').$$

If the same point be on the axis of y , then $x'' = 0$, and the equation becomes

$$y - y' = -\frac{y'' - y'}{x'}(x - x').$$

22. COR. 2. If the point coincide with the origin, then x'' and y'' being each $= 0$,

$$y - y' = \frac{y'}{x'}(x - x');$$

$$\text{or } y = \frac{y'}{x'}x$$

as it ought to be. Art. 9.

23. COR. 3. We have taken x' , y' and x'' , y'' positive : but it is easy to deduce the form of the equation when the signs of these co-ordinates are different.

24. *To find the equation to a straight line which is drawn through a given point, parallel to a given straight line.*

Let the co-ordinates of the given point be x' , y' , then if the equation to the given line be

$$y = ax + b,$$

the equation to the required line will be of the form

$$y - y' = A (x - x'),$$

in which the quantity A is to be determined.

Now since the two lines are by hypothesis parallel, their inclination to the axes must be the same;

$$\therefore A = a;$$

and the equation required is

$$y - y' = a (x - x').$$

The form of this equation remains the same, whatever be the inclination of the axes.

25. COR. Hence the equation to a straight line, parallel to the straight line whose equation is

$$y = ax + b,$$

$$\text{is } y = ax + b'.$$

If the two straight lines be represented, according to Art. 15, by the equations,

$$my + nx = mn, \text{ and } m'y + n'x = m'n';$$

then putting these under the form

$$y = -\frac{n}{m}x + n \text{ and } y = -\frac{n'}{m'}x + n';$$

it is evident that when the lines are parallel,

$$\frac{n}{m} = \frac{n'}{m'}, \text{ or } nm' = mn'.$$

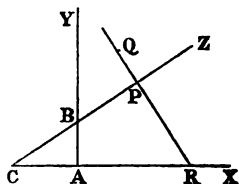
26. *To find the equation to a straight line which is drawn, through a given point, perpendicular to a given straight line.*

Let Q be the given point, x', y' its co-ordinates; CZ the given line, and $y = ax + b$ its equation.

Through Q draw QR perpendicular to CZ , meeting AX in R .

Then because QR passes through a given point, its equation will be (12) of the form

$$y - y' = A(x - x') \dots \dots \dots (1),$$



but $A = \tan QRX = -\tan QRA$

$$= -\cot ZCX$$

$$= \therefore -\frac{1}{a};$$

therefore by substitution in (1),

$$y - y' = -\frac{1}{a}(x - x'),$$

which is the equation required.

27. Cor. Hence the equation to a straight line, perpendicular to the straight line whose equation is

$$y = ax + b,$$

$$\text{is } y = -\frac{1}{a}x + b'.$$

It may be shewn, as in Art. 25, that when two straight lines, whose equations are

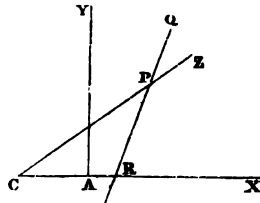
$$my + nx = mn, \text{ and } m'y + n'x = m'n',$$

are perpendicular to each other,

$$\frac{n}{m} = -\frac{m'}{n'} \text{ or } mm' = -nn'.$$

28. *To find the equation to a straight line drawn through a given point, and forming a given angle with a given straight line.*

Let Q be the given point, CZ the given straight line,



and QR an indefinite straight line drawn through Q , and making a given angle QPZ or CPR with CZ : to find the equation to QR .

Let $\tan CPR = m$, $\tan QRX = A$;

then retaining the same notation as in the last proposition, the equation to QR will be of the form

$$y - y' = A(x - x') \dots \dots \dots (1),$$

where A is to be found.

It is evident that

$$\angle CPR = PRX - PCX;$$

$$\therefore \tan CPR = \tan (PRX - PCX)$$

$$= \frac{\tan PRX - \tan PCX}{1 + \tan PRX \tan PCX};$$

but $\tan PCX = a$; and $\tan PRX = A$;

therefore by substitution,

$$m = \frac{A - a}{1 + Aa};$$

$$\therefore m(1 + Aa) = A - a;$$

$$\therefore A = \frac{a + m}{1 - am};$$

therefore by substitution in (1) we have

$$y - y' = \frac{a + m}{1 - am} (x - x'),$$

which is the equation required.

29. COR. 1. When the lines are parallel, $m = 0$; therefore, the equation becomes

$$y - y' = a(x - x'),$$

as in Art. 24.

30. COR. 2. When the lines are perpendicular, m is infinitely great; therefore, since a in the numerator, and 1 in the denominator, vanish with respect to m , the equation becomes

$$\begin{aligned} y - y' &= \frac{m}{-am} (x - x') \\ &= \therefore -\frac{1}{a} (x - x'), \end{aligned}$$

as in Art. 26.

31. COR. 3. It appears from the preceding investigation that if two straight lines, whose equations are

$$y = ax + b,$$

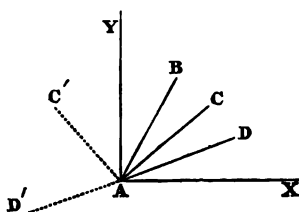
$$y = a'x + b',$$

intersect each other in an angle whose trigonometrical tangent is m ,

$$m = \frac{a - a'}{1 + aa'}.$$

32. *To find the equation to the straight line which bisects the angle contained by two given straight lines.*

Let BAD be the given angle, and AC the line which



bisects it.

The origin and direction of the axes being arbitrary, let them be AX , AY , originating at A , and at right angles to each other.

Then since the lines AB , AD , AC pass through the origin, they may be represented by the equations

$$y = ax,$$

$$y = a'x,$$

$$\text{and } y = mx,$$

in which it is the object of the problem to determine m .

By Art. 31,

$$\tan DAC = \frac{m - a'}{1 + ma'},$$

$$\text{and } \tan BAC = \frac{a - m}{1 + am};$$

but these values are equal, because AC bisects the angle BAD ;

$$\therefore \frac{m - a'}{1 + ma'} = \frac{a - m}{1 + am};$$

$$\therefore (a + a')m^2 + 2(1 - aa')m - (a + a') = 0,$$

$$\text{or } m^2 + 2\frac{1 - aa'}{a + a'}m - 1 = 0.$$

From this quadratic two real values of m may be obtained; whence it follows that there are two straight lines which resolve the problem, one bisecting the angle itself, and the other its supplement. Also, since the product of these values of $m = -1$, the straight lines are at right angles to each other.

Produce DA to D' , and from A draw AC' at right angles to AC , then AC and AC' are the two bisecting lines; and if the equation to the one be

$$y = mx,$$

that to the other will be

$$y = -\frac{1}{m}x.$$

Observation. To avoid repetition, we shall in future designate the point, *whose co-ordinates are* x' and y' , as the point (x', y') ; and the straight line *whose equation is*

$$y = ax + b,$$

as the straight line

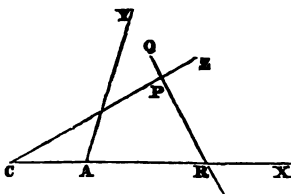
$$y = ax + b.$$

33. We have hitherto considered the straight line as referred, for the sake of simplicity, to rectangular co-ordinates. When oblique co-ordinates are employed, the form of the equation will be the same as before, so long as the indeterminate coefficient a in that equation remains unaltered. It is evident therefore that, whether rectangular or oblique co-ordinates be made use of, the form of the equation to a straight line will be the same in the following cases :

1. When it passes through one given point, or through two given points.
2. When it passes through a given point, and is parallel to a given line.

In other cases the form of the equation will be different.

34. We proceed to find the equation to a straight line passing through a given point, and perpendicular to a given line; the axes being oblique.



Let $Q (x', y')$ be the given point, and CZ , $y = ax + b$ the given line.

Then if the indefinite line QR be drawn through Q perpendicular to CZ , its equation will be of the form

$$y - y' = a'(x - x') \dots \dots \dots (1),$$

where a' is to be determined.

Let the angle $YAX = \gamma$, the angle $ZCX = \theta$, and the angle $QRX = \phi$;

$$\text{then } \tan \phi = -\tan PRC = -\cot PCR$$

$$= \therefore -\frac{1}{\tan \theta}.$$

$$\begin{aligned} \text{Now } a &= \frac{\sin \theta}{\sin (\gamma - \theta)} \\ &= \frac{\sin \theta}{\sin \gamma \cos \theta - \cos \gamma \sin \theta}; \end{aligned}$$

or dividing numerator and denominator by $\cos \theta$,

$$a = \frac{\tan \theta}{\sin \gamma - \cos \gamma \tan \theta};$$

$$\therefore \tan \theta = \frac{a \sin \gamma}{1 + a \cos \gamma}.$$

In like manner, since

$$a' = \frac{\sin \phi}{\sin (\gamma - \phi)}$$

$$\tan \phi = \frac{a' \sin \gamma}{1 + a' \cos \gamma};$$

$$\text{but } \tan \phi = -\frac{1}{\tan \theta};$$

$$\therefore \frac{a' \sin \gamma}{1 + a' \cos \gamma} = - \frac{1 + a \cos \gamma}{a \sin \gamma};$$

$$\therefore a' = - \frac{1 + a \cos \gamma}{a + \cos \gamma};$$

hence by substitution in (1) we have

$$y - y' = - \frac{1 + a \cos \gamma}{a + \cos \gamma} (x - x'),$$

the equation required.

35. If ω denote the angle of intersection of two straight lines

$$y = ax + b$$

$$y = a'x + b',$$

it may be shewn by a method similar to that employed in Art. 28, that

$$\tan \omega = \frac{(a - a') \sin \gamma}{1 + (a + a') \cos \gamma + aa'}.$$

Hence since $\sin^2 \omega = \cos^2 \omega \cdot \tan^2 \omega$

$$\text{and } \cos^2 \omega = \frac{1}{\sec^2 \omega} = \frac{1}{1 + \tan^2 \omega},$$

it may easily be proved that

$$\sin \omega = \frac{a - a'}{\sqrt{1 + a^2} \cdot \sqrt{1 + a'^2}}.$$

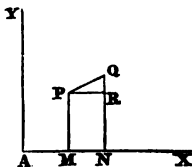
$$\text{and } \cos \omega = \pm \frac{1 + aa'}{\sqrt{1 + a^2} \cdot \sqrt{1 + a'^2}}.$$

SECTION III.

ON THE APPLICATION OF THE EQUATION TO THE
STRAIGHT LINE.

36. *To express the distance between any two points in terms of their co-ordinates.*

Let P, Q , be the given points, x', y' the co-ordinates of



the former, x'', y'' those of the latter, and suppose the axes to be rectangular.

Through P draw PR parallel to AX , meeting the ordinate of Q in the point R .

$$\begin{aligned}
 \text{Then } PQ^2 &= PR^2 + RQ^2 \\
 &= (AN - AM)^2 + (NQ - MP)^2 \\
 &= (x'' - x')^2 + (y'' - y')^2; \\
 \therefore PQ &= \sqrt{\{(x'' - x')^2 + (y'' - y')^2\}};
 \end{aligned}$$

which is the distance required.

If the points P, Q do not lie within the angle YAX , the co-ordinates x', y' and x'', y'' must have their proper signs given to them.

37. **COR.** If either point, P for example, coincide with the origin, then $x' = 0$ and $y' = 0$;

$$\therefore AQ = \sqrt{(x''^2 + y''^2)}.$$

38. *When the axes are oblique, the distance PQ may be thus found.*

In the last figure conceive YA to be inclined to AX at an angle γ , then PM being parallel to AY and PR to AX , we have

$$\cos PRQ = \frac{PR^2 + RQ^2 - PQ^2}{2PR \cdot RQ};$$

$$\text{but } PRQ = \pi - \gamma;$$

$$\therefore \cos PRQ = -\cos \gamma;$$

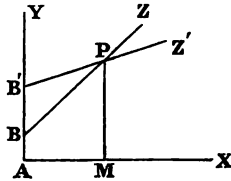
$$\therefore PQ^2 = PR^2 + RQ^2 + 2PR \cdot RQ \cos \gamma,$$

$$= (x'' - x')^2 + (y'' - y')^2 + 2(x'' - x')(y'' - y') \cos \gamma.$$

39. *To find the co-ordinates of the point in which two straight lines intersect each other.*

$$\text{Let } y = ax + b \dots \dots \dots (1),$$

$$\text{and } y = a'x + b' \dots \dots \dots (2),$$



be the equations to two lines BZ , $B'Z'$, intersecting in P ; and let x' , y' denote the co-ordinates AM and MP of that point.

Then because the point P is common to the two lines BZ , $B'Z'$, its co-ordinates will satisfy each of the above equations. We have, therefore,

$$y' = ax' + b,$$

$$\text{and } y' = a'x' + b';$$

whence, eliminating y' from these equations,

$$(a - a')x' = -(b - b');$$

$$\therefore x' = -\frac{b - b'}{a - a'};$$

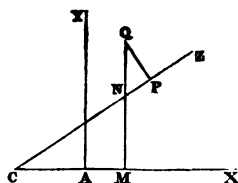
also, eliminating x' from the same equations,

$$y' = \frac{ab' - ba'}{a - a'},$$

whence x' and y' , the co-ordinates required, are known.

40. It appears from the preceding proposition, that the intersection of any two straight lines is analogous to the elimination of x and y from their equations. As this remark is applicable to any two lines whatever, situated in the same plane, it may be stated as a general principle, that "*elimination between any two equations corresponds to the intersection of their loci.*"

41. *To find the length of the perpendicular let fall from a given point upon a given straight line.*



Let Q be the given point, CZ the given straight line; from Q draw PQ perpendicular to CZ and QM perpen-

dicular to CX . Then, the axes being supposed rectangular, if x' , y' be the co-ordinates of Q , and $y = ax + b$ the equation to CZ , it is evident that since $AM = x'$,

$$MN \text{ must } = ax' + b.$$

$$\text{Now } QP = QN \sin QNP$$

$$= QN \cos NCM$$

$$= (QM - MN) \cos NCM$$

$$= (y' - ax' - b) \cos NCM;$$

$$\begin{aligned} \text{but } \cos NCM &= \frac{1}{\sec NCM} = \pm \frac{1}{\sqrt{1 + \tan^2 NCM}} \\ &= \pm \frac{1}{\sqrt{1 + a^2}}; \end{aligned}$$

therefore by substitution

$$QP = \pm \frac{y' - ax' - b}{\sqrt{1 + a^2}},$$

the upper or lower sign being taken, according as the point Q is above or below CZ .

42. COR. 1. If the given line BZ pass through the origin, $b = 0$, and

$$QP = \pm \frac{y' - ax'}{\sqrt{a^2 + 1}}.$$

43. COR. 2. If the given point Q coincide with the origin, then x' and y' each $= 0$, and

$$QP = \mp \frac{b}{\sqrt{a^2 + 1}},$$

according as the line is situated below, or above, AX .

44. The value of QP may be found by a method more strictly analytical, but more operose, than the preceding.

The principal steps are as follows:

the equation to CZ is $y = ax + b$;

and that to QR is $y - y' = -\frac{1}{a}(x - x')$,

by Art. 27.

Now P the point of intersection being common to these two lines, its co-ordinates will satisfy these equations;

$$\therefore y'' = ax'' + b \dots \dots \dots (1)$$

$$\text{and } y'' - y' = -\frac{1}{a}(x'' - x') \dots \dots \dots (2);$$

but $PQ = \sqrt{\{(x'' - x')^2 + (y'' - y')^2\}}$, Art. 36;

therefore substituting for $(y'' - y')^2$ its value $\frac{(x'' - x')^2}{a^2}$ deduced from (2), and eliminating y'' between (1) and (2), we find

$$x'' - x' = \frac{ay' - ab - a^2x'}{a^2 + 1};$$

$$\therefore QP = \pm \frac{y' - ax' - b}{\sqrt{a^2 + 1}},$$

as before.

45. To find the value of QP , when the axes are oblique.

Conceive YA in the last figure to be inclined to AX , and draw QM parallel to it.

$$\begin{aligned} \text{Then } QP &= QN \sin QNP \\ &= (MQ - MN) \sin QNP \\ &= (y' - ax' - b) \sin QNP. \end{aligned}$$

Now adopting the notation of Art. 11, it is evident that

$$\sin QNP = \sin \rho, y;$$

$$\therefore QP = (y' - ax - b) \sin \rho, y.$$

Cor. When the axes are rectangular,

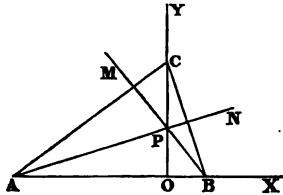
$$\sin \rho, y = \cos \rho, x = \frac{1}{\sec \rho, x} = \pm \frac{1}{\sqrt{a^2 + 1}};$$

$$\therefore QP = \pm \frac{y' - ax' - b}{\sqrt{a^2 + 1}},$$

as above.

46. The following propositions will furnish additional illustrations of the application of the equation to the straight line.

(1) *To prove that the perpendiculars let fall from the angles of a triangle on the opposite sides intersect one another in the same point.*



Let ABC be a triangle, from the angles of which the perpendiculars AN , BM , CO are let fall on the sides BC , CA , AB ; to prove that these perpendiculars meet in the same point.

Let $OB = m$, $OA = m'$, $OC = n$, and let OX , OY be assumed as the rectangular axes to which the figure is referred.

Then the equation to AN , which is drawn through a point $(-m', 0)$, is

$$y = a(x + m'),$$

but AN is perpendicular to BC ,

$$\therefore a = -\frac{1}{\tan CBX} (21) = \cot CBO = \frac{m}{n},$$

therefore the equation to AN is

$$y = \frac{m}{n}(x + m') \dots \dots \dots (1).$$

Again, the equation to BM , drawn through a point $(+m, 0)$, is

$$y = a'(x - m),$$

but BM being perpendicular to CA ,

$$a' = -\frac{1}{\tan CAB} = -\cot CAB = \frac{-m'}{n};$$

therefore the equation to BM is

$$y = \frac{-m'}{n}(x - m) \dots \dots \dots (2).$$

Let AN , BM be now supposed to meet OY , in which case $x = 0$ in these two equations;

$$\text{Then in (1), } y = \frac{mm'}{n},$$

$$\text{and in (2), } y = \frac{mm'}{n},$$

and these values of (y) being identical, the lines AN , BM meet OC in the same point; in other words, the three perpendiculars AN , BM , CO intersect one another in the same point P , as was to be proved.

This proposition admits of a very simple proof by means of the equations in Art. 16.

Let $OP = n'$, then, the same notation remaining,

the equation to BC is $my + nx = mn$,

and that to AN is $m'y + n'x = m'n'$.

But these lines being perpendicular to each other,

$$mm' = -nn', \quad \text{Art. 27.....(1).}$$

Again,

the equation to BP is $my + n'x = mn'$,

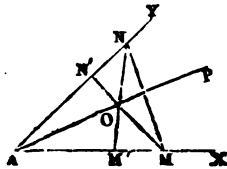
and that to AC is $m'y + nx = m'n$;

but from (1) $mm' = -nn'$;

$\therefore BP$ is perpendicular to AC ,

which was to be proved.

(2) *To prove that the straight line which joins the middle points of any two sides of a triangle is parallel to the third side.*



Let AMN be any triangle, M' , N' the middle points of AM , AN , and conceive $M'N'$ to be drawn.

Resuming the notation of Art. 16,

the equation to MN is $my + nx = mn$(1),

and that to $M'N'$ $m'y + n'x = m'n'$(2).

But by hypothesis, $m' = \frac{1}{2}m$ and $n' = \frac{1}{2}n$;
therefore by substitution, equation (2) becomes

$$my + nx = \frac{1}{2}mn;$$

hence by Art. 25,

$M'N'$ is parallel to MN .

(3) *To find the equation to the line AP, which joins the origin A, and the point O, in which MN' , NM' meet each other.*

The same construction and notation remaining,

the equation to MN' is $my + n'x = mn'$,

and that to NM' is $m'y + nx = m'n$.

The equation to the line passing through the origin and the point of intersection of these two lines will be of the form

$$y = ax,$$

in which a is to be determined.

The co-ordinates of the point of intersection being the same for the three lines, we have, on multiplying the first by $m'n$, the second by mn' , and then subtracting the products,

$$mm'(n - n')y + nn'(m' - m)x = 0,$$

$$\text{or } y = -\frac{nn'}{mm'} \cdot \frac{m' - m}{n - n'} x,$$

which is evidently the equation required.

COR. 1. Let M' and N' be the middle points of AM , AN .

Then $m' = \frac{1}{2}m$ and $n' = \frac{1}{2}n$;

therefore by substitution, the equation to AP becomes

$$y = \frac{n'}{m'} x.$$

Let $x = m'$ or AM' ;

$$\text{then } y = n' \text{ or } \frac{1}{2} AN;$$

therefore AP passes through the middle point of MN .

Hence *the three lines drawn from the angles of any triangle to the middle points of the opposite sides intersect in the same point.*

COR. 2. Let MN' , NM' be supposed to bisect the angles M , N .

Then by EUC. VI. 3,

$$\frac{n'}{n - n'} = \frac{m}{MN}$$

and

$$\frac{m - m'}{m'} = \frac{MN}{n};$$

$$\text{therefore } \frac{n'}{m'} \cdot \frac{m - m'}{n - n'} = \frac{m}{n};$$

and by substitution, the equation to AP becomes

$$y = x;$$

hence AP bisects the angle A .

Wherefore, *the lines bisecting the three angles of any triangle meet each other in the same point.*

CHAPTER III.

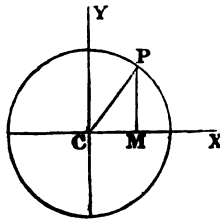
ON THE CIRCLE.

SECTION I.

ON THE EQUATION TO THE CIRCLE.

47. *To find the equation to the circle, when referred to rectangular co-ordinates.*

Let C be the centre of the circle, whose equation it is



required to find.

The position of the axes being arbitrary, we shall assume

(1) The *centre* as the origin.

Through C draw CX and CY at right angles to each other; in the circumference take any point P , from which let fall the perpendicular PM on CX , and let $CM = x$, $MP = y$, $CP = r$.

Then, in the right-angled triangle CMP ,

$$CM^2 + MP^2 = CP^2;$$

or, by substitution,

$$x^2 + y^2 = r^2;$$

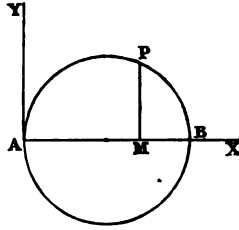
and, as the same relation may be shewn to subsist between

the co-ordinates of every other point in the circumference, the equation required is

$$x^2 + y^2 = r^2 \dots \dots \dots (1).$$

- (2) Let the *extremity of any diameter* be assumed as the origin.

Let A be the origin, and let $AM = x$, $MP = y$. Then,



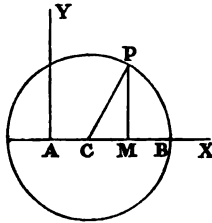
by a well-known property of the circle,

$$MP^2 = AM \cdot MB,$$

$$\begin{aligned} \text{or } y^2 &= x(2r - x) \\ &= 2rx - x^2 \dots \dots \dots (2), \end{aligned}$$

which is the equation required.

- (3) Let *any point in the axis of x* be assumed as the origin.



Then, if $AC = x'$, we have, as before,

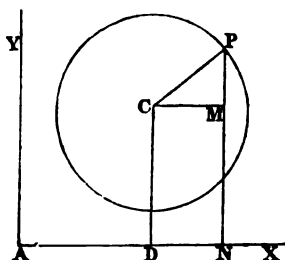
$$CM^2 + MP^2 = CP^2,$$

$$\text{or } (x - x')^2 + y^2 = r^2 \dots \dots \dots (3).$$

In like manner, if the origin be on the axis of y , the equation will be

$$x^2 + (y - y')^2 = r^2.$$

(4) *Let any point whatever, within or without the circle, be assumed as the origin.*



Let the point A , without the circle, be the origin, and let the co-ordinates AD , DC of the centre be denoted by x' and y' ; also let $AN = x$, $NP = y$, then

$$CM^2 + MP^2 = CP^2,$$

$$\text{or } (AN - AD)^2 + (NP - CD)^2 = CP^2;$$

$$\therefore (x - x')^2 + (y - y')^2 = r^2 \dots \dots \dots (4),$$

which is the equation required.

48. The equation to the circle, when the axes are rectangular, and originate at any point, is, as has just been shewn,

$$(x - x')^2 + (y - y')^2 = r^2,$$

which, expanded, becomes

$$y^2 + x^2 - 2y' \cdot y - 2x' \cdot x + y'^2 + x'^2 - r^2 = 0 \dots \dots \dots (1).$$

Now the general equation of the second degree between two variables is

$$ay^2 + bxy + cx^2 + Ay + Bx + C = 0 \dots \dots \dots (2).$$

Comparing the corresponding terms of (1), and (2), we have

$$a = 1, \quad b = 0, \quad c = 1,$$

whence it follows that, *the general form of the equation to the circle, when referred to rectangular co-ordinates, is*

$$y^2 + x^2 + Ay + Bx + C = 0,$$

in which the term involving xy is wanting.

49. *To find the position and magnitude of the circle, which is the locus of the equation*

$$y^2 + x^2 + Ay + Bx + C = 0.$$

Comparing it with equation (1) in the last Article,

$$A = -2y', \quad B = -2x', \quad C = x'^2 + y'^2 - r^2;$$

$$\therefore y' = -\frac{A}{2}, \quad x' = -\frac{B}{2},$$

whence, as the co-ordinates of the centre are known, the *position* of the circle is determined.

Again,

$$\begin{aligned} r^2 &= x'^2 + y'^2 - C \\ &= \frac{A^2}{4} + \frac{B^2}{4} - C; \end{aligned}$$

$$\therefore r = \frac{1}{2} \sqrt{(A^2 + B^2 - 4C)};$$

therefore, the *magnitude* of the circle is also known.

Ex. To find the position and magnitude of the circle, which is the locus of the equation

$$y^2 + x^2 = 6x - 8y.$$

By transposition,

$$y^2 + x^2 + 8y - 6x = 0 \dots \dots \dots (1),$$

and comparing this with the equation

$$y^2 + x^2 - 2y'y - 2x'x + y'^2 + x'^2 - r^2 = 0 \dots \dots \dots (2),$$

we have

$$2y' = -8; \quad \therefore y' = -4,$$

$$2x' = 6; \therefore x' = 3.$$

But

$$\begin{aligned} y'^2 + x'^2 &= r^2 = \therefore 16 + 9 \\ &= 25; \end{aligned}$$

$$\therefore r = 5.$$

Since $r^2 = x'^2 + y'^2$, it is plain that the origin must be on the circumference. Conceive therefore AY , AX , to be two rectangular axes; in AX take $AB = 3$, from B draw BC at right angles to AX , and on the opposite side of it, $= 4$; then a circle, described from the point C as centre, and with CA as radius, will be the circle required.

50. When the axes are inclined at any angle whatever, the general equation to the circle is (Art. 38)

$$(x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos \alpha, y = r^2,$$

the expression $\cos \alpha$, y denoting the cosine of the angle contained by the axes of x and y .

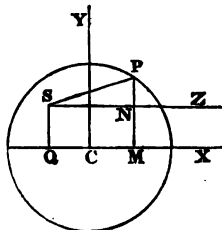
51. It thence appears that the equation to the circle is in all cases of the second degree, and that the simplicity of its form depends on the position and inclination of the axes of co-ordinates. The equations generally used are

$$1. \quad x^2 + y^2 = r^2,$$

$$\text{and } 2. \quad y^2 = 2rx - x^2.$$

52. To find the polar equation to the circle.

Let C be the centre, and P any point in the circumference



of the circle, whose polar equation is required.

Assume any point S as the pole, and through it draw SZ parallel to CX , meeting MP in N , and let the ordinate of S meet CX in the point Q .

Let $SP = \rho$, angle $PSZ = \omega$.

Let the rectangular co-ordinates of S be $-\alpha, \beta$; and let those of P be x, y .

Then in general

$$x^2 + y^2 = r^2 \dots\dots\dots(1),$$

$$\text{but } x = CM = SN - QC = \rho \cos \omega + \alpha,$$

$$y = PM = SQ + PN = \beta + \rho \sin \omega,$$

substituting these values of x and y in equation (1), we have

$$(\alpha + \rho \cos \omega)^2 + (\beta + \rho \sin \omega)^2 = r^2,$$

and expanding, and reducing the result,

$$\rho^2 + 2(\alpha \cos \omega + \beta \sin \omega) \rho + \alpha^2 + \beta^2 - r^2 = 0,$$

which is the polar equation required.

When the point S is without the circle, the polar equation will be

$$\rho^2 - 2(\alpha \cos \omega + \beta \sin \omega) \rho + \alpha^2 + \beta^2 - r^2 = 0.$$

53. COR. Let PS be conceived to meet the circumference in p .

Then SP, Sp are the two values of ρ .

But by the theory of equations

$$SP \cdot Sp = \alpha^2 + \beta^2 - r^2;$$

now this value of $SP \cdot Sp$, being independent of the angle ω , is the same for all lines drawn through the point S ;

Hence, *if two chords intersect each other, within or without a circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

SECTION II.

ON THE APPLICATION OF THE EQUATION TO THE CIRCLE.

54. *To determine the intersection of a straight line with a circle.*

Let the circle whose equation is

$$x^2 + y^2 = r^2 \dots\dots\dots(1),$$

be intersected by a straight line

$$y = ax + b \dots\dots\dots(2),$$

and let x' , y' be the co-ordinates at the points of intersection.

Then since these must satisfy (1) and (2), we have

$$x'^2 + y'^2 = r^2,$$

$$\text{and } y' = ax' + b;$$

from the latter of these, we have $x' = \frac{y' - b}{a}$; therefore by substitution in the former,

$$y'^2 + \left(\frac{y' - b}{a}\right)^2 = r^2;$$

$$\therefore a^2 y'^2 + y'^2 - 2by' + b^2 = a^2 r^2;$$

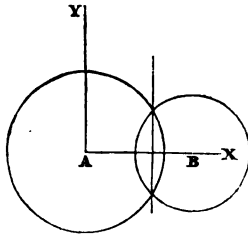
$$\therefore y'^2 - \frac{2b}{a^2 + 1} y' + \frac{b^2 - a^2 r^2}{a^2 + 1} = 0,$$

from which quadratic two values of y' may be obtained; and therefore, also, two values of x' may be found by substitution in either of the above equations (1) and (2).

If the two values of y' be *equal*, the two points of section coincide, and the straight line touches the circle; and if the values of y' be *imaginary*, the straight line falls entirely without the circle.

Hence a straight line cannot cut a circle in more than two points.

55. *If two circles cut each other, to find the co-ordinates of the points of intersection.*



Let A and B be the centres of the two circles: join A, B ; assume AB as the axis of x , and AY , perpendicular to AB , as the axis of y .

If r and r' be the radii of the circles, and AB be denoted by x' , then the equation to the circle A will be

$$y^2 + x^2 = r^2 \dots \dots \dots (1),$$

and that to the circle B ,

$$y^2 + (x - x')^2 = r'^2 \dots \dots \dots (2).$$

When the circles cut each other, the values of x and y in their equations will be identical;

Therefore subtracting the second from the first,

$$2xx' - x'^2 = r^2 - r'^2;$$

$$\therefore x = \frac{r^2 - r'^2 + x'^2}{2x'}.$$

Substituting this value of x in equation (1),

$$\begin{aligned} y^2 &= r^2 - \left(\frac{r^2 - r'^2 + x'^2}{2x'} \right)^2 \\ &= \frac{1}{4x'^2} \{ 4r^2 x'^2 - (r^2 - r'^2 + x'^2)^2 \} \\ &= \frac{1}{4x'^2} \{ (2rx' + r^2 - r'^2 + x'^2) (2rx' - r^2 + r'^2 - x'^2) \}; \end{aligned}$$

but

$$2rx' + r^2 - r'^2 + x'^2 = (r + x')^2 - r'^2 = (r + x' + r') (r + x' - r'),$$

and

$$2rx' - r^2 + r'^2 - x'^2 = r'^2 - (r - x')^2 = (r + r' - x') (r - r' + x');$$

hence, by substitution,

$$y = \pm \frac{1}{2x'} \sqrt{\{ (r + r' + x') (r + x' - r') (r' + x' - r) (r + r' - x') \}},$$

the co-ordinates therefore of the points of intersection are known.

56. COR. 1. Since for each value of x there are two equal values of y , with opposite signs, it follows that the line joining the points of section is bisected at right angles by the line joining the centres.

57. COR. 2. That the circles may cut each other, these values of y must be real; that is, the quantity under the radical must be positive. This will evidently be the case, when the factors are all positive, or when two of them are positive, and two negative: or, since the first is necessarily positive, when any two of the others are negative, and the remaining one positive. But by supposing any one of these three factors negative, the other two will be found to be positive:

thus let $r + x' - r'$ be negative ;

$$\therefore r + x' < r' ;$$

$\therefore r$ and x' must each be $< r'$;

$$\therefore r - r' + x' \text{ and } r' - x' + r,$$

are positive.

Whence it appears that all the factors under the radical must be positive, in order that an intersection of the circles may take place.

58. COR. 3. When the circles touch each other, the values of $y = 0$,

$$\therefore r + x' - r' = 0,$$

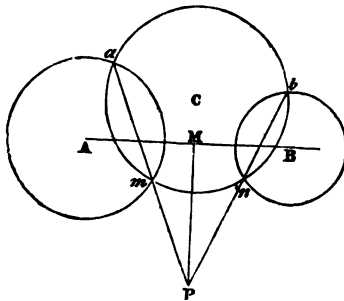
$$r' + x' - r = 0,$$

$$\text{and } r + r' - x' = 0 ;$$

$$\therefore x' = r \pm r',$$

hence, *two circles touch each other, externally or internally, when the line joining their centres is equal to the sum or difference of their radii.*

59. *Any two circles being cut by a third, to find where the chords joining the points of intersection will meet.*



Let the two circles whose centres are A, B and radii r, r' , be cut by a third, whose centre is C and radius r'' , in the points a, m and b, n : draw am, bn meeting in P : to find the position of P .

The axes being supposed rectangular, let A be the origin, and AB the axis of x : and let the co-ordinates of C be x'' , y'' .

Then the equation to the circle A is $y^2 + x^2 = r^2$(1),

that to $B \dots y^2 + (x - x')^2 = r'^2$(2),

and that to $C \dots (y - y'')^2 + (x - x'')^2 = r''^2$...(3).

When C cuts A , the values of x and y in their equations will be identical,

therefore subtracting (3) from (1),

$$2xx'' + 2yy'' = x''^2 + y''^2 + r^2 - r''^2.$$

This being the equation to a straight line, it can be no other than the equation to the chord am which joins the points of contact.

In like manner, the equation to the chord bn is

$$2xx'' + 2yy'' - 2xx' = r'^2 - r''^2 - x'^2 + x''^2 + y''^2.$$

When the chords, am , bn meet, the values of x and y in their equations are the same;

therefore subtracting the former from the latter, and dividing by $2x'$, we have

$$x = \frac{r^2 - r'^2 + x'^2}{2x'},$$

which is the equation to a straight line parallel to the axis of y .

This value of x being constant, and independent of r'' , x'' , y'' , it follows that *whatever be the position and magnitude of the circle C, the point of intersection of all chords am , bn will lie in the line MP, which is at right angles to AB.*

60. COR. 1. Since $BM = AB - AM = x' - x$

$$= \frac{x'^2 - r^2 + r'^2}{2x'};$$

$$\therefore BM - AM = \frac{r'^2 - r^2}{x'} = \frac{r'^2 - r^2}{BM + AM};$$

$$\therefore BM^2 - AM^2 = r'^2 - r^2.$$

61. COR. 2. If from P , any point in MP , there be drawn PQ , Pq tangents to A , B respectively, then PQ will $= Pq$.

$$\text{For } PQ^2 = Pa \cdot Pm = Pb \cdot Pn = Pq^2;$$

$$\therefore PQ = Pq.$$

62. COR. 3. If the circles A and B cut each other, it is evident that the line joining the points of intersection will coincide with MP ; Art. 56.

Hence, *when three circles cut each other, the chords which join the points of intersection of each two of them will meet in the same point.*

63. *To find the equation to a straight line touching a circle in a given point.*

Let x', y' be the co-ordinates of a given point, and x'', y'' those of any other point in the circumference, near the first.

Then the equation to the straight line drawn through these two points, and cutting the circumference is (20),

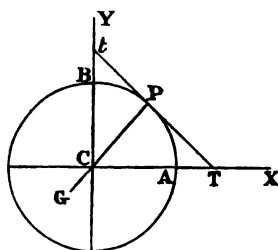
$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \dots \dots \dots (1).$$

But these two points being in the circumference, we have

$$y'^2 + x'^2 = r^2,$$

$$\text{and } y''^2 + x''^2 = r^2;$$

$$\therefore y''^2 - y'^2 + x''^2 - x'^2 = 0;$$



$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'};$$

and by substitution in (1),

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x').$$

Let the points (x'', y'') and (x', y') be now supposed to coincide, and then $x'' = x'$, $y'' = y'$, and the secant becomes a tangent, therefore the equation to the tangent is

$$y - y' = -\frac{2x'}{2y'}(x - x'),$$

$$\text{or, } y - y' = -\frac{x'}{y'}(x - x').$$

64. **Cor.** This equation may be reduced to a simpler form; for multiplying both sides by y' ,

$$yy' - y'^2 = -xx' + x'^2;$$

$$\therefore yy' + xx' = y'^2 + x'^2$$

$$= \therefore r^2.$$

Observation. In deducing the equation to the tangent we have employed the foregoing method, because it is applicable to the Conic Sections in general; but in the particular case of the circle, the problem admits of a simpler solution.

The equation to a straight line meeting the circle in a given point (x', y') is

$$y - y' = A(x - x') \dots \dots \dots (1).$$

Now the equation to a line joining the centre and the given point is

$$y = \frac{y'}{x'} x \dots \dots \dots (2).$$

when these two straight lines are at right angles, the former becomes a tangent; but in this case $A = \frac{-x'}{y'}$ Art. 27, therefore by substitution in (1), the equation to the tangent becomes

$$y - y' = \frac{-x'}{y'} (x - x'),$$

as before.

DEF. If a straight line be drawn from the point of contact at right angles to the tangent, it is called the *normal*.

65. *To find the equation to the normal.*

Let PT be the tangent at P , from which point draw PG at right angles to PT .

Then because PG is drawn through the point (x', y') , its equation is of the form

$$y - y' = A(x - x'),$$

but PG being, by hypothesis, perpendicular to PT , whose equation is

$$y - y' = -\frac{x'}{y'}(x - x'),$$

$$A = \frac{y'}{x'}(27);$$

therefore the equation required is

$$y - y' = \frac{y'}{x'}(x - x'),$$

or multiplying by x' ,

$$yx' - y'x' = xy' - y'x';$$

$$\therefore y = \frac{y'}{x'}x,$$

the equation to a line passing through the origin, which in this case is the centre. Hence the normals all pass through the centre; and conversely, a straight line drawn from the centre to the point of contact is at right angles to the tangent.

66. *To draw a tangent to a circle from a given point without it.*

Let x'', y'' be the co-ordinates of the given point, and x', y' the unknown co-ordinates of the point of contact; then the equation to the tangent being, in general,

$$yy' + xx' = r^2,$$

and the point (x'', y'') being by hypothesis a point in the tangent, we have

$$y''y' + x''x' = r^2 \dots \dots \dots (1),$$

also, the point of contact being a point in the circumference,

$$y'^2 + x'^2 = r^2 \dots \dots \dots (2).$$

Hence by means of these two equations the co-ordinates x', y' required may be determined.

The reducing equation, which results from elimination between (1) and (2), will manifestly be of the second degree; hence there are *two* points of contact, in other words, two tangents may be drawn to a circle, from a given point without it.

This problem affords an illustration of the principle stated in Art. 39, that "*elimination between any two equations corresponds to the intersection of their loci:*"

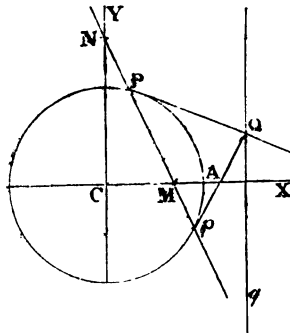
The two reducing equations are

$$y'^2 + x'^2 = r^2 \dots \dots \dots (1),$$

$$y' y'' + x' x'' = r^2 \dots \dots \dots (2).$$

The first of these, considered by itself, represents a circle, which is the locus of all the points whose co-ordinates satisfy equation (1); and the second, considered by itself, represents a straight line, which is the locus of all the points whose co-ordinates satisfy equation (2). These loci being constructed, and referred to the same axes, it is plain that the points in which they intersect will be the points of contact required, because the co-ordinates of those points are the particular values of x' and y' which satisfy both equations.

Now the locus of the first equation is the given circle; let



its centre C be assumed as the origin of the axes CX , CY . The straight line, which is the locus of the second equation, is constructed as in Art. 14.

$$\text{Let } y' = 0; \therefore x' = \frac{r^2}{x''},$$

$$x' = 0; \therefore y' = \frac{r^2}{y''}.$$

In CX take $CM = \frac{r^2}{x'}$, and in CY take $CN = \frac{r^2}{y'}$, join M, N and let MN cut the circle in the points P, p ; these will be the points of contact required.

67. COR. 1. Because the straight line Pp determines, by its intersection with the circle, the points of contact, it follows that the equation

$$y'y' + x''x' = r^2,$$

in which x' and y' are the variables, is the equation to the line joining the points of contact.

68. COR. 2. Hence also may be proved the following theorem.

If from the several points of a straight line, given in position, pairs of tangents be drawn to a circle, the straight lines joining the points of contact, in each case, will all pass through the same point.

For let C be the centre of the given circle, (Fig. p. 51.) Qq the line given in position, and from any point $Q (x'', y'')$ draw the two tangents QP, Qp ; from C let fall the perpendicular CX on Qq , and draw CY at right angles to CX ; then CX and CY being assumed as axes, the equation to the line Pp will be, by the last Cor.

$$y'y'' + x'x'' = r^2,$$

in which x' and y' are the co-ordinates of P .

Now let Pp meet CX , then $y' = 0$; and

$$\therefore x' = \frac{r^2}{x''} = CM.$$

Since this value of CM is independent of y'' , it will remain the same for all points whose abscissas are $= x''$, that is, for all points in the given line Qq , as was to be proved.

CHAPTER IV.

ON THE TRANSFORMATION OF CO-ORDINATES.

69. THE equations to the straight line and circle remain always of the same degree; that is, the former is always a simple equation, and the latter a quadratic: but it has been shewn, that the greater or less simplicity in the form of these equations depends on the position of the straight line and circle, with respect to the axes of co-ordinates, and on the inclination of the axes themselves.

Thus for instance, the general equation to the straight line

$$y = ax + b,$$

becomes $y = ax$,

when the straight line passes through the origin: and in each case the coefficient a has a value more or less simple, according as the axes are rectangular or oblique.

Again, the most general equation to the circle is obtained by referring it to oblique axes, originating at any point within or without the circle: the equation is then simplified, first by supposing the axes rectangular, and then by assigning different positions to the origin.

In like manner, the most general equation to the curves which are denominated the CONIC SECTIONS may be successively simplified, by a suitable disposition of the origin and direction of the axes of co-ordinates.

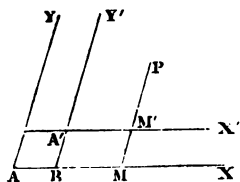
DEF. The operation of changing the origin, or the direction of the axes, or of changing at once both their origin and direction, is called the *Transformation of co-ordinates*.

The object of this Chapter is to resolve the following Problem: *having given the equation to a curve, when referred to one system of axes, to find the equation to the same curve, when referred to any other system of axes.*

The problem presents these three cases; first, when the origin of the axes is changed, their direction remaining the same: secondly, when the direction alone of the axes is changed, their origin remaining the same: and thirdly, when both the origin and the direction of the axes are changed.

70. CASE 1. *Let the origin alone of the axes be changed, their inclination remaining the same.*

Let AX, AY be the old axes, $A'X', A'Y'$ the new ones; the latter being supposed to be parallel to the former.



Let P be any point, AM, MP , and $A'M', M'P$ its co-ordinates when referred to the old and the new axes respectively: produce $Y'A'$ to meet AX in B . Then AB, BA' are the co-ordinates of the new origin.

Let

$$AB = \alpha, BA' = \beta, AM = x, MP = y \text{ and } A'M' = x', M'P = y.$$

$$\text{Then } AM = AB + BM = AB + A'M';$$

$$\therefore x = \alpha + x' \dots \dots \dots (1).$$

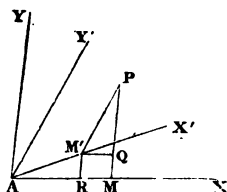
$$\text{Also } MP = MM' + M'P = BA' + M'P;$$

$$\therefore y = \beta + y' \dots \dots \dots (2).$$

If these values of x and y be substituted in the equation to the proposed curve, referred to AX, AY , we shall obtain the equation to the same curve, when referred to the new axes $A'X', A'Y'$.

71. CASE 2. *Let the inclination alone of the axes be changed, the origin remaining the same.*

Let AX, AY be the old axes, $A'X', A'Y'$ the new ones :



Let P be any point, draw PM, PM' parallel to $AY, A'Y'$ respectively, and from M' draw $M'Q$ parallel to AX , and MR parallel to AY .

Then AM, MP and $AM', M'P$ are the co-ordinates of P , when referred to the old and the new axes respectively.

Let γ denote the inclination of the old axes to each other, and θ, θ' the respective inclinations of $A'X', A'Y'$ to AX .

Then $YAX = \gamma, X'AX = \theta, Y'AX = \theta'$;

also let $AM = x, MP = y; AM' = x', M'P = y'$.

The object of the problem is to express x and y in terms of x' and y' , and of the quantities γ, θ, θ' .

Now $x = AM = AR + RM = AR + M'Q$.

$$\text{But } AR = AM' \frac{\sin AM'R}{\sin ARM'} = AM' \frac{\sin YAX'}{\sin YAX} = x' \frac{\sin (\gamma - \theta)}{\sin \gamma},$$

$$M'Q = M'P \frac{\sin M'PQ}{\sin M'QP} = M'P \frac{\sin YAY'}{\sin YAX} = y' \frac{\sin (\gamma - \theta')}{\sin \gamma};$$

therefore by substitution,

$$x = x' \frac{\sin(\gamma - \theta)}{\sin \gamma} + y' \frac{\sin(\gamma - \theta')}{\sin \gamma} \dots\dots\dots(1).$$

$$\text{Again, } y = MP = MQ + QP = RM' + QP.$$

$$\text{But } RM' = AM' \frac{\sin RAM'}{\sin ARM'} = AM' \frac{\sin X'AX}{\sin YAX} = x' \frac{\sin \theta}{\sin \gamma};$$

$$QP = M'P \frac{\sin PM'Q}{\sin PQM'} = M'P \frac{\sin Y'AX}{\sin YAX} = y' \frac{\sin \theta'}{\sin \gamma};$$

therefore by substitution,

$$y = x' \frac{\sin \theta}{\sin \gamma} + y' \frac{\sin \theta'}{\sin \gamma} \dots\dots\dots(2).$$

From (1) and (2) therefore, we have for the required values of x and y ,

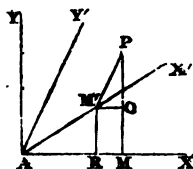
$$x = \frac{1}{\sin \gamma} \{x' \sin(\gamma - \theta) + y' \sin(\gamma - \theta')\},$$

$$y = \frac{1}{\sin \gamma} \{x' \sin \theta + y' \sin \theta'\}.$$

These are the most general formulas: we will now consider the following particular cases.

72. *Let the old axes be rectangular, and the new ones oblique.*

Then a similar construction being made,



$$AM = AR + M'Q$$

$$= AM' \cos X'AX + M'P \cos PM'Q,$$

$$\text{or } x = x' \cos \theta + y' \cos \theta'.$$

Similarly,

$$y = x' \sin \theta + y' \sin \theta'.$$

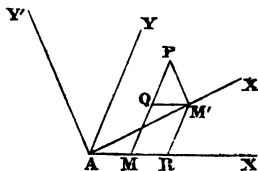
These results are also deducible from the general expressions in the last Article, observing that

$$\sin \gamma = \sin \frac{\pi}{2} = 1,$$

$$\sin (\gamma - \theta) = \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta,$$

$$\text{and } \sin (\gamma - \theta') = \sin \left(\frac{\pi}{2} - \theta' \right) = \cos \theta'.$$

73. *Let the old axes be oblique, and the new ones rectangular.*



$$\text{Here } AM = AR - M'Q;$$

$$\text{but } AR = AM' \frac{\sin AM'R}{\sin ARM'} = AM' \frac{\sin YAX'}{\sin YAX} = x' \frac{\sin (\gamma - \theta)}{\sin \gamma},$$

$$\text{and } M'Q = M'P \frac{\sin M'PQ}{\sin PQM'} = M'P \frac{\sin YAY'}{\sin YAX} = y' \frac{\cos (\gamma - \theta)}{\sin \gamma};$$

$$\therefore x = \frac{1}{\sin \gamma} \{x' \sin (\gamma - \theta) - y' \cos (\gamma - \theta)\}.$$

Similarly,

$$y = \frac{1}{\sin \gamma} \{x' \sin \theta + y' \cos \theta\}.$$

These results are deducible from the general expressions, observing that, since

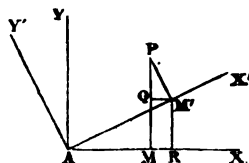
$$\theta' = \frac{\pi}{2} + \theta,$$

$$\sin(\gamma - \theta') = \sin\left\{\gamma - \frac{\pi}{2} - \theta\right\} = -\sin\left\{\frac{\pi}{2} - (\gamma - \theta)\right\}$$

$$= -\cos(\gamma - \theta);$$

$$\text{and } \sin \theta' = \cos \theta.$$

74. *Let both the old and the new axes be rectangular.*



$$\text{Then } AM = AR - QM'$$

$$= AM' \cos X'AX - PM' \cos PM'Q$$

$$= AM' \cos X'AX - PM' \sin X'AX,$$

or

$$x = x' \cos \theta - y' \sin \theta.$$

Similarly,

$$y = x' \sin \theta + y' \cos \theta.$$

These results, as in the two former cases, may be immediately deduced from the general expressions.

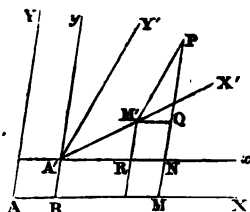
$$\text{For } \sin \gamma = \sin \frac{\pi}{2} = 1,$$

$$\sin(\gamma - \theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta,$$

$$\sin(\gamma - \theta') = \sin\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + \theta\right)\right\} = -\sin \theta,$$

$$\sin \theta' = \sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta.$$

75. CASE 8. *Let both the origin and inclination of the axes be changed.*



Let AX , AY be the old axes,

$A'X'$, $A'Y'$ the new ones.

Through A' draw $A'x$, $A'y$ parallel to AX , AY , and complete the figure. Then AB , BA' will be the co-ordinates of the new origin.

Let $AB = \alpha$, $BA' = \beta$.

Then $AM = AB + BM$

$$= AB + A'R + M'Q.$$

Therefore substituting for $A'R$ and $M'Q$ the values found in Art. 71, we have

$$x = \alpha + \frac{1}{\sin \gamma} \{x' \sin (\gamma - \theta) + y' \sin (\gamma - \theta')\},$$

and similarly

$$y = \beta + \frac{1}{\sin \gamma} \{x' \sin \theta + y' \sin \theta'\}.$$

76. When the origin is changed in the several cases considered in Articles 72, 73 and 74, we have only to add the abscissa of the new origin to the value of x , and the ordinate of the new origin to the value of y .

77. In the preceding investigations we have supposed, for the sake of simplicity, that the signs of α and β are both positive, and that the new axis $A'X'$ is situated above

the old one AX . If α or β , or both of them, be negative, or if the axis AX' be below AX , the foregoing formulas will undergo certain modifications, which the student will have no difficulty in discovering.

When it is proposed to refer a curve to axes, whose position and direction are simpler than those of the original ones, the quantities α , β and θ , θ' are to be considered as known.

But when the transformation is made for the purpose of simplifying the general equation, by taking away one or more of its terms, then α , β and θ , θ' are to be considered as *indeterminate constants*, whose values may be deduced from the conditions of the question.

In all cases γ , the inclination of the original axes, is supposed to be a given quantity.

78. *Observation.* The notation explained in Art. 11 may be used with great advantage, in the formulas for the transformation of co-ordinates.

By the adoption, for example, of this notation, the general expressions in Art. 71, become

$$x = \frac{1}{\sin \alpha, y} \{x' \sin \alpha', y + y' \sin \gamma', y\}$$

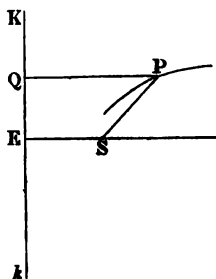
$$y = \frac{1}{\sin \alpha, y} \{x' \sin \alpha', x + y' \sin \gamma', x\};$$

where we perceive at a glance the relation of the different angles, which enter into these formulas.

CONIC SECTIONS.

79. DEF. *A Conic Section is the locus of a point, whose distances from a fixed point, and a straight line given in position, are to each other in a constant ratio.*

Thus, let S be a fixed point, Kk a straight line given in



position, P any point; join P, S , and let fall the perpendicular PQ upon Kk ; then, if PS be always taken to PQ in the same constant ratio, the locus of P will be a Conic Section.

The fixed point S is called the *focus*, and the straight line Kk , given in position, the *directrix*.

80. The particular species of the Conic Section will depend upon the constant ratio of $PS : PQ$, which may be either a ratio of *equality*, or of lesser, or greater, *inequality*.

(1). Let $PS = PQ$.

Then the locus of P is called the *Parabola*.

(2). Let $PS < PQ$.

Then the locus of P is called the *Ellipse*.

(3). Let $PS > PQ$.

Then the locus of P is called the *Hyperbola*.

ON THE PARABOLA.

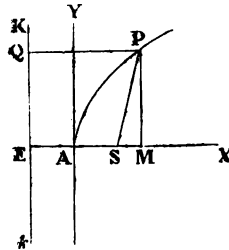
CHAPTER I.

ON THE PARABOLA REFERRED TO ITS AXIS.

81. *To find the equation to the parabola.*

DEF. The parabola is the locus of a point, whose distance from the focus is always equal to its perpendicular distance from the directrix.

Let S be the focus, Kk the directrix, P any point in the



parabola; through S draw the indefinite line ESX perpendicular to the directrix; from P let fall the perpendiculars PM and PQ on EX , Kk respectively, and join P , S .

Then, if ES be bisected in A , the point A is, agreeably to the definition, a point in the parabola; from A draw AY at

right angles to AX , and assume AX and AY as the rectangular axes, to which the parabola is to be referred.

Let $AM = x$, $MP = y$, and $AS = m$.

Then $SP^2 = PM^2 + MS^2 = y^2 + (x - m)^2 \dots\dots (1)$,

but $SP^2 = PQ^2 = EM^2 = (EA + AM)^2$
 $= (m + x)^2 \dots\dots\dots (2)$.

Whence, equating these two values of SP^2 ,

$$y^2 + (x - m)^2 = (x + m)^2,$$

$$\therefore y^2 = 4mx,$$

which is the equation required.

82. *To determine the figure of the parabola, from its equation.*

The same axes being employed, the equation to the parabola is

$$y^2 = 4mx,$$

$$\text{or } y = \pm 2\sqrt{mx}.$$

Let $x = 0$, then $y = 0$;

therefore the curve passes through the origin A .

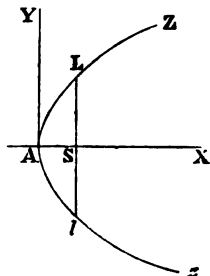
Let x be supposed to have any *positive* value.

Then, for each assumed value of x , there are *two* equal values of y , with contrary signs; as x increases, the values of y increase; and when x is taken indefinitely great, the values of y will also become indefinitely great.

Let x be now supposed to have any *negative* value.

Then the values of y being in this case imaginary, it is plain that no part of the curve can lie to the *left* of A .

The parabola consists, therefore, of two infinite branches



AZ , Az , situated to the *right* of the point A , and symmetrically placed with respect to the straight line AX .

The point A is called the *vertex*, and the line AX the *axis*, of the parabola.

83. COR. 1. The parabola can have but one focus, and one directrix.

84. COR. 2. To find the value of SL , the ordinate passing through the focus.

At the point L , $x = AS = m$,

$$\therefore y^2 = 4m^2;$$

$$\therefore y = \pm 2m = SL \text{ or } Sl.$$

The double ordinate Ll , passing through the focus, is called the *principal parameter*, or *latus rectum*, of the parabola,

85. COR. 3. Hence, if P be any point in the parabola, Fig. p. 62,

$$PM^2 = Ll \cdot AM;$$

that is, *the square of the ordinate is equal to the latus rectum multiplied into the corresponding abscissa*.

86. To find the intersection of a straight line with a parabola.

Let the equation to the proposed straight line be

$$y = ax + \beta \dots \dots \dots (1).$$

Then the co-ordinates of the point or points of intersection with the parabola will be determined by combining this equation with the equation

$$y^2 = 4mx \dots\dots\dots (2).$$

Substituting, then, in (2) the value of x derived from (1), we have

$$y^2 = 4m \cdot \frac{y - \beta}{a},$$

$$\text{or } y^2 - \frac{4m}{a}y + \frac{4m\beta}{a} = 0.$$

This quadratic gives two values of y , which, substituted in (1), furnish two corresponding values of x ; therefore the co-ordinates required may be determined.

When the two roots of the quadratic are *equal*, the points of section coincide, and the straight line will then *touch* the parabola; and when the two roots are *imaginary*, the straight line falls entirely without the parabola.

Hence it appears, that a straight line cannot cut a parabola in more than two points.

DEF. That part of the straight line contained within the parabola is called a *chord*; when it passes through the focus, it is then called the *focal chord*.

87. *To find the equation to a straight line, that touches a parabola in a given point.**

Let x', y' be the co-ordinates of the given point, and x'', y'' those of any other point in the parabola, near the first.

Then the equation to the straight line, drawn through these two points, and cutting the parabola, is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x') \dots\dots\dots (1).$$

* The method of investigation is the same with that already employed in the case of the circle, Art. 63.

But these points being in the parabola, we have

$$y'^2 = 4mx',$$

$$\text{and } y''^2 = 4mx'';$$

$$\therefore y''^2 - y'^2 = 4m(x'' - x');$$

$$\therefore \frac{y'' - y'}{x'' - x'} = \frac{4m}{y'' + y'};$$

therefore equation (1), becomes, by substitution,

$$y - y' = \frac{4m}{y'' + y'}(x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y') , then

$$x'' = x', \quad y'' = y',$$

and the *secant*, or cutting line, will become a *tangent*.

Hence the equation to the tangent is

$$y - y' = \frac{2m}{y'}(x - x'),$$

in which x', y' are the co-ordinates of the point of contact, and x, y , the variable co-ordinates of any point whatever in the tangent.

88. COR. The equation to the tangent may be presented under a more commodious form; for, multiplying each side by y' , we have

$$yy' - y'^2 = 2mx - 2mx',$$

$$\text{but } y'^2 = 4mx';$$

$$\begin{aligned} \therefore yy' &= 2mx + 4mx' - 2mx' \\ &= 2m(x + x'), \end{aligned}$$

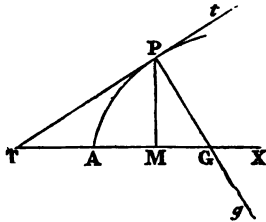
which is the equation most commonly used.

89. *To find the intersection of the tangent with the axis.*

In the equation

$$yy' = 2m(x + x').$$

Let $y = 0$, as at T , then $x + x' = 0$, or $x' = -x$.



Now x represents AT , and x' , AM ;

$$\therefore AT = AM;$$

the negative sign merely implying, that AT must be measured in the contrary direction to AM .

90. COR. 1. Hence $MT = 2MA$.

DEF. The line MT , intercepted between the foot of the ordinate and the point where the tangent meets the axis, is called the *subtangent*.

It appears, therefore, that the subtangent is equal to twice the abscissa.

91. COR. 2. Hence is derived a simple method of drawing a tangent to a parabola at a given point.

Let P be the given point, and AM , MP its co-ordinates; in MA produced take $AT = AM$, join T , P , then TP touches the parabola in P .

DEF. The straight line which is drawn from the point of contact at right angles to the tangent is called the *normal*.

92. *To find the equation to the normal.*

Let TP touch the parabola in P , and from this point draw Pg at right angles to PT .

Then, since Pg is at right angles to PT , whose equation is

$$y - y' = \frac{2m}{y'}(x - x'), \quad (87),$$

the equation to Pg will be (27),

$$y - y' = -\frac{y'}{2m}(x - x').$$

93. *To find the intersection of the normal with the axis.*

When the normal cuts the axis, as at G , then $y = 0$;

$$\therefore -y' = -\frac{y'}{2m}(x - x'),$$

$$\therefore x - x' = 2m;$$

that is, $AG - AM$, or $MG = 2m$.

DEF. The line MG , intercepted between the foot of the ordinate, and the point where the normal cuts the axis, is called the *subnormal*.

Hence it appears, that the *subnormal is equal to half the latus rectum*.

We have considered the normal as an indefinite straight line Pg , but it is customary to give that name to the finite straight line PG , intercepted between the point of contact, and the point in which Pg cuts the axis.

94. *To draw a tangent to a parabola from a given point without it.*

Let $Q(x'', y'')$ be the given point, and let x', y' be the *unknown* co-ordinates of the point of contact.

Hence, take AM in the opposite direction to $AX = x''$, and in AY take $AN = 2m \frac{x''}{y''}$; join M, N , and let MN cut the parabola in the points P, p ; these will be the points of contact required.

95. COR. 1. Since the straight line, which has just been constructed, determines, by its intersection with the parabola, the points of contact, it follows that the equation

$$y''y' = 2m(x' + x''),$$

in which x' and y' are variable, is the equation to the indefinite straight line Pp joining the points of contact.

96. COR. 2. Because AM is independent of y'' , it will remain the same for all points whose abscissas are $= x''$; that is, for all points in the indefinite straight line Qq , drawn through Q perpendicular to AX . Hence the following theorem, analogous to that in Art. 68;

If from the several points of a straight line perpendicular to the axis, pairs of tangents be drawn to a parabola, the chords joining the points of contact, in each case, will all pass through the same point.

97. COR. 3. If the given straight line be the directrix, then $x'' = -m$, therefore $AM = m = AS$; hence *all the chords will in this case pass through the focus*. Wherefore the equation to the focal chord of contact is

$$y''y' = 2m(x' - m).$$

CHAPTER II.

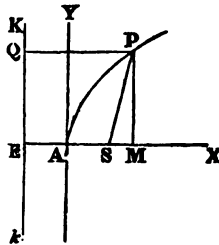
ON THE PARABOLA REFERRED TO THE FOCUS.

98. *To find the distance of any point in the parabola from the focus.*

Let x, y be the co-ordinates of any point P , and r the required distance SP ; then, in general, the distance between any two points (x, y) , and (x', y')

$$= \sqrt{\{(x - x')^2 + (y - y')^2\}}.$$

Now the focus being a point in AX , $y' = 0$, and $x' = AS = m$; therefore by substitution



$$\begin{aligned} r &= \sqrt{\{(x - m)^2 + y^2\}} \\ &= \sqrt{\{(x - m)^2 + 4mx\}} \\ &= \sqrt{(x + m)^2} \\ &= \therefore x + m \end{aligned}$$

as was required.

This value of SP may be determined at once, by referring to the directrix ; for

$$SP = PQ = AE + AM = m + x.$$

99. *To find the polar equation to the parabola, the focus being the pole.*

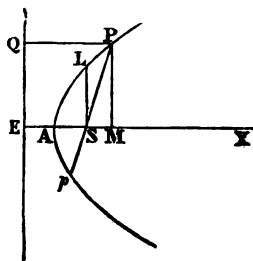
Let P be any point, whose co-ordinates are AM , MP , and let $SP = r$, angle $ASP = \omega$.

Then (98) $r = m + x$,

$$\text{but } x = AM = AS + SM$$

$$= m + SP \cos PSM$$

$$= m - r \cos \omega ;$$



therefore, by substitution,

$$r = 2m - r \cos \omega ;$$

$$\therefore r = \frac{2m}{1 + \cos \omega} \dots\dots\dots(1),$$

$$\text{or } = \frac{m}{\cos^2 \frac{\omega}{2}} \dots\dots\dots(2).$$

100. COR. 1. If PS be produced to meet the parabola in p , and Sp be denoted by r' , then since $\angle SPp = \pi - \omega$,

$$r' = \frac{2m}{1 - \cos \omega},$$

$$\text{or } = \frac{m}{\sin^2 \frac{\omega}{2}}.$$

101. COR. 2. Hence

$$\frac{1}{r} + \frac{1}{r'} = \frac{1 + \cos \omega}{2m} + \frac{1 - \cos \omega}{2m} = \frac{2}{2m},$$

$$\text{or } \frac{1}{SP} + \frac{1}{Sp} = \frac{2}{SL},$$

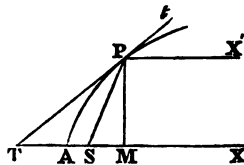
that is, the *principal semi-parameter* is an *harmonic mean* between the segments of any chord drawn through the focus.

102. COR. 3. Since $\frac{1}{r} + \frac{1}{r'} = \frac{r + r'}{rr'}$ and also $= \frac{2}{2m}$;

$$\therefore rr' = m(r + r'),$$

that is, $SP \cdot Sp = m \cdot Pp$.

103. If from the point of contact two straight lines be drawn, one to the focus, and the other parallel to the axis, they will make equal angles with the tangent.



Let TP be a tangent, from P draw PX' parallel to AX , and join P, S ; the angle $\angle tPX' = \angle SPT$.

For since $AT = AM$,

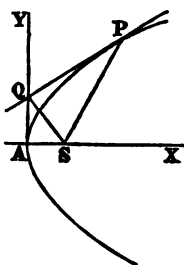
$$ST = SA + AT = m + x = \therefore SP \dots\dots (97);$$

therefore angle $SPT = \text{angle } STP = \text{angle } tPX'$,

because PX' is parallel to AX .

104. *The tangent at any point, and the perpendicular let fall upon it from the focus, intersect AY in the same point.*

Let PQ be a tangent at P , from S let fall the perpendicular



SQ upon it, meeting it in the point Q ; to prove that Q is a point in AY .

The equation to PQ is

$$y = \frac{2m}{y'} (x + x') \dots\dots\dots (1),$$

and the equation to SQ , drawn from S ($x = m$, $y = 0$), and perpendicular to PQ , is

$$y = -\frac{y'}{2m} (x - m) \dots\dots\dots (2).$$

Now when PQ and SQ meet AY , x must = 0 in both equations, we therefore have from (1),

$$y = 2m \frac{x'}{y'} = 2m \cdot \frac{y'^2}{4my'} = \frac{y'}{2}; \text{ because } x' = \frac{y'^2}{4m},$$

and from (2) $y = \frac{y'}{2}$;

and as these values of y are identical, PQ and SQ meet AY in the same point.

105. COR. Conceive PQ to meet XA in T , then because SQT is a right angle,

$$ST \cdot SA = SQ^2, \quad \text{EUC. VI. 8. Cor.}$$

or since $ST = SP$,

$$SP \cdot SA = SQ^2.$$

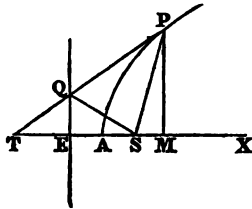
106. *If two lines be drawn from the focus, one to the point of contact, and the other to the point in which the tangent meets the directrix, they will be perpendicular to each other.*

Let the tangent at $P(x', y')$ meet the directrix EQ in Q , then drawing SP , SQ , it is required to prove that SP is perpendicular to SQ .

The equation to the tangent is, in general, (88)

$$y = \frac{2m}{y'}(x + x'),$$

now when it meets the directrix, $x = -m$;



$$\therefore y \text{ or } EQ = \frac{2m}{y'}(x' - m).$$

Again, the equation to SQ is

$$y = -\tan QSE(x - m),$$

$$= -\frac{QE}{2m}(x - m), \text{ or substituting for } QE \text{ its value,}$$

$$= -\frac{x' - m}{y'}(x - m) \dots \dots \dots (1).$$

Also, the equation to SP is

$$\begin{aligned} y &= \tan PSX (x - m), \\ &= \frac{PM}{MS} (x - m), \\ &= \therefore \frac{y'}{x' - m} (x - m) \dots \dots \dots (2); \end{aligned}$$

therefore comparing the coefficients in (1) and (2), it follows, (Art. 27) that SP is perpendicular to SQ .

The proposition may be at once proved, by employing the equation to the focal chord of contact. For that equation being (97),

$$y' = \frac{2m}{y''} (x' - m),$$

and the equation to SQ being

$$\begin{aligned} y &= -\frac{QE}{ES} (x - m), \\ &= -\frac{y''}{2m} (x - m), \text{ since } QE = y'', \end{aligned}$$

it follows that SQ is perpendicular to SP .

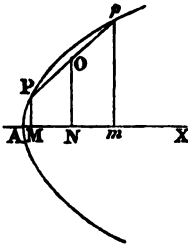
CHAPTER III.

ON THE PARABOLA REFERRED TO ANY DIAMETER.

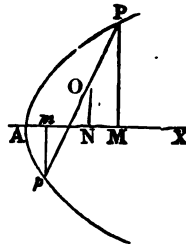
107. *To find the locus of the middle points of any number of parallel chords.*

Let Pp be any chord, O its middle point; from the points

(1)



(2)



P, O, p , let fall the perpendiculars PM, ON, pm , on the axis AX ; then if the equation to Pp be

$$y = ax + \beta,$$

the equation containing the values of y at the points, P, p will be

$$y^2 - \frac{4m}{a}y + \frac{4m\beta}{a} = 0. \quad (\text{Art. 86.})$$

Now, since in any quadratic equation the coefficient of the second term, with its proper sign, is equal to the sum of the roots with their signs changed,

$$\frac{4m}{a} = PM + pm.$$

But O being the middle point of Pp ,

$$ON = \frac{PM + pm}{2}, \text{ in Fig. (1),}$$

$$= \therefore \frac{2m}{a}.$$

Now since m is constant, and a remains the same for all chords parallel to Pp (25), this value of ON is invariable; in other words, if ON be denoted by y , the equation to the middle points of any number of parallel chords is

$$y = \text{constant};$$

therefore the locus required is a straight line, parallel to the axis AX .

DEF. 1. The straight line which has just been shewn to bisect any number of parallel chords is called a *diameter*.

DEF. 2. Each half of the chord, so bisected, is called an *ordinate* to the diameter bisecting it.

108. COR. 1. The diameters of the parabola are parallel to the axis, and intersect the curve only in one point.

The truth of the first part of the corollary is evident from the proposition; that of the second may be thus proved:

If c be any constant quantity, the equation to any diameter is $y = c$; therefore the intersection of the diameter with the parabola will be determined by combining this equation with the equation

$$y^2 = 4mx;$$

$$\text{we therefore have } c^2 = 4mx;$$

$$\therefore x = \frac{c^2}{4m}.$$

Hence there can be only *one* point of intersection.

109. COR. 2. If the equation to any chord be

$$y = ax + b \dots \dots \dots (1),$$

the equation to a diameter, passing through any point (x', y') , and bisecting that chord, will be (107) $y' = \frac{2m}{a} \dots \dots \dots (2)$.

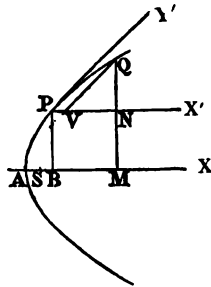
Conversely, since $a = \frac{2m}{y'}$, the ordinate to a diameter passing through the point (x', y') will have for its equation

$$y = \frac{2m}{y'} x + b' \dots \dots \dots (3).$$

110. COR. 3. Comparing equation (3) with the equation to a tangent at the point (x', y') , Art. 87, it appears, *that the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.*

111. *To find the equation to the parabola, when it is referred to any diameter and the tangent at its vertex, as axes.*

Let PX' be any diameter, and PY' the tangent at its ver-



tex, take Q any point in the parabola, and draw QM perpendicular to AX , and QV parallel to PY' ; and from P let fall the perpendicular PB on AX .

Assume $AM = x$, $MQ = y$; $PV = x'$, $VQ = y'$;

also let $AB = a$, $BP = \beta$, and the angle $Y'PX' = \theta$.

The object now is, to determine the relation between x' and y' .

In general (81), $y^2 = 4mx \dots \dots \dots (1)$.

$$\text{But } y = MN + NQ = \beta + y' \sin \theta,$$

$$\text{and } x = AB + BM = \alpha + x' + y' \cos \theta^* ;$$

therefore by substitution in (1),

$$(\beta + y' \sin \theta)^2 = 4m(\alpha + x' + y' \cos \theta),$$

or developing, and arranging the result according to the dimensions of y' ,

$$y'^2 \sin^2 \theta + 2(\beta \sin \theta - 2m \cos \theta) y' + \beta^2 = 4m\alpha + 4mx' \dots\dots(2),$$

but (α, β) being a point in the parabola, we have by equation (1),

$$\beta^2 = 4m\alpha,$$

$$\text{also, } \tan \theta = \frac{2m}{\beta} (87) ;$$

$$\therefore \beta \sin \theta - 2m \cos \theta = 0,$$

$\therefore$ the term involving y' vanishes; hence equation (2) becomes

$$y'^2 = \frac{4m}{\sin^2 \theta} x' \dots\dots\dots(3),$$

$$\text{but angle } ASP + 2\theta = \pi ; \therefore \theta = \frac{\pi}{2} - \frac{1}{2}ASP ;$$

$$\therefore \frac{m}{\sin^2 \theta} = \frac{m}{\cos^2 \frac{1}{2}ASP} = SP (99) ;$$

therefore, by substitution in (3),

$$y'^2 = 4SP \cdot x',$$

which is the equation required.

DEF. When any diameter and the tangent at its vertex are assumed as axes of co-ordinates, they are termed *conjugate axes*.

112. COR. 1. Hence, suppressing the accents, and denoting SP by m , the equation to the parabola, when referred to any conjugate axes whatever, is $y^2 = 4mx$.

* This is a particular case of the formulas in Art. 72 for the transformation of co-ordinates from a rectangular to an oblique system, the new axis of x remaining parallel to the old one. See Art. 76.

DEF. The double ordinate passing through the focus is called the *parameter* of the corresponding diameter; hence

113. COR. 2. The parameter of any diameter is equal to four times the distance of its vertex from the focus.

114. COR. 3. Hence also, *the square of the ordinate to any diameter is equal to the parameter multiplied into the corresponding abscissa.* See Art. 85.

115. The equation to the tangent, when the parabola is referred to any diameter, is of the same form as before, namely,

$$y = \frac{2m}{y'}(x + x'),$$

the coefficient $\frac{2m}{y'}$ denoting, in this case, the ratio of the sines of the angles which the tangent makes with the axes of x and y , Art. 10.

116. COR. 1. When the tangent meets the axis of x , then $y = 0$;

$$\therefore x = -x';$$

that is, *the subtangent is bisected by the curve, whether the co-ordinates are rectangular or oblique.* See Art. 90.

117. COR. 2. Hence also, whatever be the inclination of the axes, the equation to an ordinate to a diameter passing through any point (x', y') , is

$$y = \frac{2m}{y'}x + \beta.$$

See Art. 109.

118. *If from the several points of a straight line given in position, pairs of tangents be drawn to a parabola, the straight lines joining the corresponding points of contact will all pass through the same point.*

Let MN be the given straight line, and AX the axis of the parabola; through any point in AX draw a chord mn parallel to MN , let PX' be the diameter which bisects this chord, and at the vertex P apply a tangent PY' , which (110) will be parallel to MN .

Then the equation to the parabola, when referred to the oblique axes PX' and PY' , will be (111)

$$y^2 = 4mx \dots\dots\dots(1),$$

and if from any point (x'', y'') in MN a pair of tangents be drawn to the parabola, it may be shewn, precisely as in Art. (94), that the equation to the line joining the points of contact is

$$y''y' = 2m(x' + x'') \dots\dots\dots(2),$$

in which x' and y' are the variable co-ordinates of the point of contact.

Let the chord (2) cut the axis of x , then $y' = 0$, and therefore

$$x' = -x'';$$

hence the point of intersection will be the same for all points whose abscissa = x'' , that is, for all points in the given straight line MN , as was to be proved.

119. *If from the point of intersection of two tangents a diameter be drawn, it will bisect the straight line joining the points of contact.*

For the equation to an ordinate to the diameter passing through (x'', y'') , is (117)

$$y = \frac{2m}{y''}x + \beta \dots\dots\dots(1).$$

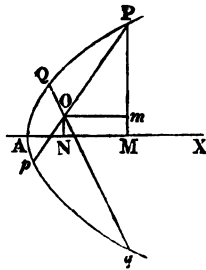
And the equation to the line joining the points of contact is

$$y' = \frac{2m}{y''}(x' + x'') \dots\dots\dots(2),$$

therefore the latter, being parallel to the former, is also an ordinate, and is consequently bisected.

120. *If through any point within or without a parabola two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will be to that contained by the segments of the other, in a constant ratio.*

Let O be any point within the parabola PAp , and let any two straight lines, given in position, be drawn through that point, so as to meet the curve in the points P, p and Q, q ; to prove that the ratio of



$OP \cdot Op : OQ \cdot Oq$ is given.

From $O(a, \beta)$ let fall the perpendiculars ON, Om upon AX and PM ; let $OP = r$, $OQ = r'$, and let θ, θ' denote the angles which r, r' make with AX .

Then if the parabola be referred to rectangular co-ordinates*,

$$y^2 = 4mx \dots \dots \dots (1).$$

$$\text{But } y = ON + Pm = \beta + r \sin \theta,$$

$$\text{and } x = AN + Om = a + r \cos \theta;$$

* When oblique co-ordinates are employed, the principle is the same, but the investigation is more operose.

therefore by substitution in (1),

$$(\beta + r \sin \theta)^2 = 4m(a + r \cos \theta);$$

$$\therefore r^2 + \frac{2(\beta \sin \theta - 2m \cos \theta)}{\sin^2 \theta} r = \frac{4ma - \beta^2}{\sin^2 \theta},$$

in which the values of r are OP , Op : therefore by the theory of equations,

$$OP \cdot Op = \frac{4ma - \beta^2}{\sin^2 \theta}.$$

In like manner

$$OQ \cdot Oq = \frac{4ma - \beta^2}{\sin^2 \theta'};$$

$$\therefore OP \cdot Op : OQ \cdot Oq :: \sin^2 \theta' : \sin^2 \theta,$$

and as θ , θ' are by hypothesis given, these rectangles are to each other in a constant ratio, as was to be proved.

121. COR. 1. Let lines parallel to Pp and Qq be conceived to touch the parabola in P' and Q' , then if S be the focus,

$$\sin^2 \theta = \cos^2 \cdot \frac{1}{2} ASP' = \frac{m}{SP'}, \text{ Art. 111.}$$

$$\text{and } \sin^2 \theta' = \frac{m}{SQ'};$$

$$\therefore OP \cdot Op : OQ \cdot Oq :: SP' : SQ'.$$

122. COR. 2. Let the point O be without the curve, and suppose the points P , p and Q , q to coincide; then OP , OQ become tangents, and

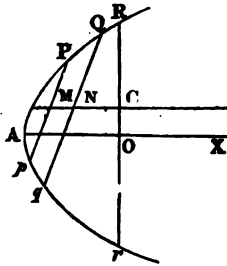
$$OP^2 : OQ^2 :: SP : SQ.$$

CHAPTER IV.

MISCELLANEOUS PROPOSITIONS.

123. *A PARABOLA being traced upon a plane, to find the position of its axis.*

Draw any two parallel chords Pp , Qq , and bisect them



in M , N ; the line joining M , N will be a diameter. Art. (107.)

In this diameter take any point C , and through C draw RCr at right angles to MN , meeting the parabola in R , r ; then if Rr be bisected in O , and AOX be drawn parallel to MN , it will evidently be the axis required.

124. *Let Pp be any chord cutting the axis in O , and let AM , Am be the respective abscissas of P and p , to prove that*

$$AM \cdot Am = AO^2.$$

Let the equation to Pp be

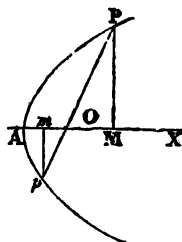
$$y = ax + b \dots \dots \dots (1),$$

then the abscissas AM , Am will be found by eliminating y between this equation and

$$y^2 = 4mx \dots\dots\dots(2);$$

we have therefore

$$(ax + b)^2 = 4mx,$$



$$\text{or } ax^2 + (2ab - 4m)x + b^2 = 0,$$

$$\text{or } x^2 + 2 \cdot \frac{ab - 2m}{a^2} x + \frac{b^2}{a^2} = 0;$$

therefore by the theory of equations, $AM \cdot Am = \frac{b^2}{a^2}$.

But at the point O , where Pp cuts the axis of x , $y = 0$;

$$\therefore x = -\frac{b}{a} = AO;$$

$$\therefore AO^2 = \frac{b^2}{a^2} = \therefore AM \cdot Am,$$

as was to be proved.

125. *In the axis AX of a given parabola to find a point O , such, that if any chord whatever POP be drawn through it, the angle PAP may be a right angle.*

Conceive the straight lines AP , Ap drawn, then, since they are at right angles to each other by hypothesis, if the equation to AP be

$$y = ax \dots\dots\dots(1),$$

that to Ap will be (27),

$$y = -\frac{1}{\alpha}x \dots\dots\dots (2).$$

Let the co-ordinates of P be x', y' , and those of p, x'', y'' ; then these co-ordinates will be determined by eliminating y between (1) and (2) and the equation to the parabola,

$$y^2 = 4mx \dots\dots\dots (3),$$

we thus have

$$\alpha^2 x'^2 = 4mx'; \quad \therefore x' = \frac{4m}{\alpha^2}, \quad \text{and} \quad \therefore y' = \frac{4m}{\alpha},$$

$$\text{also, } \frac{x''^2}{\alpha^2} = 4mx''; \quad \therefore x'' = 4m\alpha^2, \quad \text{and} \quad \therefore y'' = -4m\alpha;$$

hence the equation to Pp is (20),

$$y - \frac{4m}{\alpha} = -\frac{4m\alpha + \frac{4m}{\alpha}}{4m\alpha^2 - \frac{4m}{\alpha^2}} \left(x - \frac{4m}{\alpha^2} \right),$$

or, dividing numerator and denominator by $4m\alpha + \frac{4m}{\alpha}$,

$$y - \frac{4m}{\alpha} = -\frac{1}{\alpha - \frac{1}{\alpha}} \left(x - \frac{4m}{\alpha^2} \right).$$

Let Pp be now supposed to cut AX , as at O , then $y = 0$, and

$$\left(\alpha - \frac{1}{\alpha} \right) \frac{4m}{\alpha} = x - \frac{4m}{\alpha^2};$$

$$\therefore x \text{ or } AO = 4m,$$

whence it follows, that a point, whose distance from the vertex is equal to the *latus rectum*, has the property above enunciated.

126. *Pairs of tangents to a parabola being always supposed to intersect at right angles, to find the locus of the points of intersection.*

$$\text{Let } y = ax + \beta \dots\dots\dots(1),$$

be any straight line cutting the parabola

$$y^2 = 4mx \dots\dots\dots(2),$$

then the equation which contains the values of (y) at the points of intersection, is (86),

$$y^2 - \frac{4m}{a}y + \frac{4m\beta}{a} = 0 \dots\dots\dots(3);$$

but when these values are equal, the intersecting line becomes a tangent; hence equation (3) is in this case a perfect square, the criterion of which is, that *four times the product of the extreme terms is equal to the square of the mean*; we have therefore

$$16 \frac{m\beta}{a} = 16 \frac{m^2}{a^2};$$

$$\therefore \frac{m}{a} = \beta = y - ax, \text{ from (1);}$$

$$\therefore m = ay - a^2x;$$

$$\therefore a^2 - \frac{y}{x}a + \frac{m}{x} = 0,$$

in which equation the values of (a) are the trigonometrical tangents of the angles, which the two tangents to the parabola make with the axis; therefore the product of these values = $\frac{m}{x}$, and also = -1 , since by hypothesis the tangents are at right angles to each other;

$$\therefore \frac{m}{x} = -1,$$

$$\text{or } x = -m,$$

therefore the locus of their intersection is the directrix.

ON THE ELLIPSE.

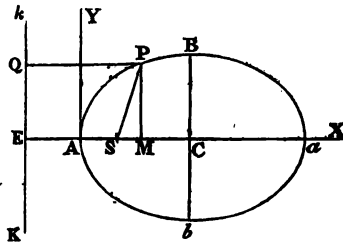
CHAPTER I.

ON THE ELLIPSE REFERRED TO ITS AXES.

127. *To find the equation to the ellipse.*

DEF. The ellipse is the locus of a point, whose distance from the focus is always less, in a given ratio, than its distance from the directrix.

Let S be the focus, Kk the directrix, P any point in the ellipse; through S draw the indefinite line ESX perpendicular to the directrix;



lar to the directrix; from P let fall the perpendiculars PM , PQ on EX , Kk respectively, and join P , S .

Let the given ratio of $PS : PQ$ be as $e : 1$, e being less than 1; then if SE be divided in A , so that $SA : AE :: e : 1$, A is a point in the ellipse.

From A draw AY at right angles to AX , and assume AX , AY as the rectangular axes, to which the ellipse is to be referred.

Let $AM = x$, $MP = y$, and $AS = m$.

Then $SP^2 = PM^2 + MS^2 = y^2 + (x - m)^2 \dots\dots\dots(1)$,

$$\begin{aligned} \text{but } SP^2 &= e^2 \cdot PQ^2 &= e^2 (AE + AM)^2 \\ &= e^2 \left(\frac{m}{e} + x \right)^2 \dots\dots\dots(2); \end{aligned}$$

therefore, equating (1) and (2),

$$y^2 + (x - m)^2 = m^2 + 2mex + e^2 x^2;$$

$$\therefore y^2 = 2m(1 + e)x - (1 - e^2)x^2$$

$$= (1 - e) \left(\frac{2m}{1 - e} x - x^2 \right),$$

or, if $\frac{m}{1 - e}$ be assumed $= a$,

$$y^2 = (1 - e^2)(2ax - x^2),$$

which is the equation required.

128. COR. 1. Bisect Aa in C , then at this point, $x = a$;

$$\therefore y^2 = (1 - e^2)a^2;$$

$$\therefore y = \pm a\sqrt{1 - e^2},$$

which is always real, since $e < 1$.

Hence if BCb be drawn through C at right angles to Aa , and CB , Cb be each taken $= a\sqrt{1 - e^2}$, the points B , b will be points in the ellipse.

129. COR. 2. Let Bb be denoted by $2b$,

$$\text{then } b = \pm a\sqrt{1 - e^2};$$

$$\therefore \sqrt{1 - e^2} = \pm \frac{b}{a};$$

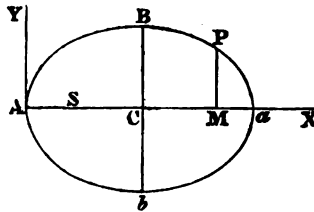
therefore, by substitution, the equation to the ellipse becomes

$$y = \pm \frac{b}{a} \sqrt{2ax - x^2} \dots \dots \dots (1).$$

DEF. The straight lines Aa and Bb , represented by $2a$ and $2b$, are called, respectively, the *major*, and the *minor*, *axis*; the points A, a, B, b , in which they meet the ellipse, are called the *vertices*; and the point C , in which they intersect each other, the *centre*.

130. To find the equation to the ellipse, when the co-ordinates originate at the centre.

Let P be any point in the ellipse, let fall the perpendicular PM on Aa , and assume $CM = x'$.



Then the equation to the ellipse, when the co-ordinates originate at A , is (129),

$$y^2 = \frac{b^2}{a^2} (2ax - x^2) \dots \dots \dots (1),$$

$$\text{but } x = AM = AC + CM$$

$$= a + x';$$

therefore, substituting this value for x , we have

$$\begin{aligned} y^2 &= \frac{b^2}{a^2} \{2a(a + x') - (a + x')^2\} \\ &= \frac{b^2}{a^2} (a^2 - x'^2) \dots \dots \dots (2), \end{aligned}$$

which is the equation required.

131. COR. 1. Suppressing the accent, which was used only to distinguish the new from the old abscissa, we have, by multiplication and transposition,

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots\dots\dots (3).$$

If each term be divided by $a^2 b^2$, we have

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \dots\dots\dots (4).$$

Of the three last forms of the equation to the ellipse, the equation marked (3) is most frequently used.

132. COR. 2. Equations (1) and (2), when translated into geometrical language, express a property of the ellipse.

For if P be any point, we have

$$2ax - x^2 = (2a - x)x = AM \cdot Ma,$$

$$\text{and } a^2 - x'^2 = (a + x')(a - x') = AM \cdot Ma;$$

$$\therefore MP^2 = \frac{BC^2}{AC^2} \cdot AM \cdot Ma,$$

$$\text{or } AM \cdot Ma : MP^2 :: AC^2 : BC^2;$$

that is, *the rectangle contained by the segments of the major axis is to the square of the ordinate, as the square of the semi-major axis is to the square of the semi-minor axis.*

133. COR. 3. If $a = b$, equations (1) and (2) become

$$y^2 = 2ax - x^2, \text{ and}$$

$$y^2 = a^2 - x^2,$$

which (47) represent a circle, whose radius is a , and which is therefore described upon the major axis Aa as a diameter. It thence appears *that the circle is a species of the ellipse.*

Hence also, if the ordinate MP of the ellipse be conceived to be produced upwards to meet the circle, described on the major axis, in the point Q ,

$$\frac{MP}{MQ} = \frac{\frac{b}{a} \sqrt{a^2 - x^2}}{\sqrt{a^2 - x^2}} = \therefore \frac{b}{a}.$$

134. *To determine the figure of the ellipse, from its equation.*

Resuming the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

we have, either

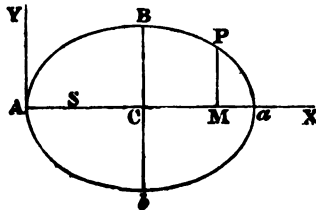
$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2} \dots \dots \dots (1),$$

$$\text{or, } x = \pm \frac{a}{b} \sqrt{b^2 - y^2} \dots \dots \dots (2),$$

First, in equation (1)

Let $x = 0$, then $y = \pm b = CB$ or Cb .

Let $y = 0$, then $x = \pm a = CA$ or Ca .



Let $x < \pm a$,

then for each value of x , there are *two* equal values of y , with contrary signs.

Let $x = \pm a$,

then $y = \pm 0$, that is, the ellipse cuts the axis of x at the points A and a .

Let $a > \pm a$,

then the quantity under the vinculum being negative, the values of y are *imaginary*, and no point of the ellipse can lie beyond a to the *right*, or A to the *left*.

It appears, therefore, that the ellipse is a continuous curve, returning into itself, and divided by the major axis Aa into two equal parts.

In the same way it may be shewn, by discussing equation (2), that the ellipse has the form just assigned to it, and that it is divided by the minor axis Bb into two equal parts.

135. COR. 1. The distance between the centre and focus
 $= CS = AC - AS = a - a(1 - e) = \therefore ae$.

The quantity ae is called the *eccentricity* of the ellipse.

136. COR. 2. To find the value of the ordinate passing through the focus.

When the ordinate passes through the focus,

$$x = ae,$$

$$\therefore y^2 = \frac{b^2}{a^2} (a^2 - a^2 e^2)$$

$$= b^2 (1 - e^2)$$

$$= \frac{b^4}{a^2} (129);$$

$$\therefore y = \pm \frac{b^2}{a} = SL \text{ or } Sl. *$$

The double ordinate Ll passing through the focus is called the *principal parameter*, or *latus rectum*.

* The straight line Ll is drawn through S , at right angles to Aa , and meeting the ellipse in L and l .

137. COR. 3. If in the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

we change x into y , and y into x , or if, in other words, we invert the position of the axes, there results

$$a^2 x^2 + b^2 y^2 = a^2 b^2,$$

the discussion of which equation would lead to the same figure of the ellipse as before. And the reason is evident, for the ellipse being symmetrically placed with respect to both its axes, it is immaterial whether we reckon the abscissas along CX , and the ordinates along CY , or the abscissas along CY , and the ordinates along CX .

138. *When the major axis is supposed to become indefinitely great, the ellipse passes into a parabola.*

The equation to the ellipse, the co-ordinates originating at the vertex, is (129)

$$\begin{aligned} y^2 &= \frac{b^2}{a^2} (2ax - x^2) \\ &= \frac{2b^2}{a} x - \frac{b^2}{a^2} x^2 \dots\dots\dots (1). \end{aligned}$$

$$\begin{aligned} \text{Now } b^2 &= a^2 - a^2 e^2 \\ &= (a + ae)(a - ae) \\ &= (a + ae) AS; \end{aligned}$$

$$\therefore \frac{b^2}{a} = (1 + e) AS;$$

and by substitution in (1),

$$y^2 = 2AS(1 + e)x - \frac{b^2}{a^2} x^2 \dots\dots\dots (2).$$

Let a be now supposed infinite, then $\frac{b^2}{a^2} = 0$, and since

$a = \frac{m}{1 - e}$ (127), we have $1 - e = 0$, or $e = 1$; therefore by substitution in (2),

$$y^2 = 4AS \cdot x,$$

which is the equation to the parabola. Art. 81.

Whence the truth of the proposition.

139. *To find the intersection of a straight line with the ellipse.*

Let the equation to the proposed line be

$$y = ax + \beta \dots \dots \dots (1).$$

Then the co-ordinates of the point or points of intersection with the ellipse, will be obtained by combining this equation with that to the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots \dots \dots (2).$$

Substituting, then, in (2) the value of x derived from (1), we have

$$\begin{aligned} a^2 y^2 + b^2 \left(\frac{y - \beta}{a} \right)^2 &= a^2 b^2; \\ \therefore (a^2 a^2 + b^2) y^2 - 2b^2 \beta y + b^2 \beta^2 &= a^2 b^2 a, \\ \therefore y^2 - \frac{2b^2 \beta}{a^2 a^2 + b^2} y + \frac{b^2 (\beta^2 - a^2 a^2)}{a^2 a^2 + b^2} &= 0. \end{aligned}$$

From this quadratic are obtained two values of y , which, substituted in (1), furnish two corresponding values of x ; therefore the co-ordinates required may be determined.

When the two roots of the quadratic are equal, the points of section coincide, and the straight line *touches* the ellipse; and when they are imaginary, the straight line falls entirely without the ellipse.

Hence it appears, that a straight line cannot cut an ellipse in more than two points.

DEF. The portion of the straight line contained within the ellipse is called a *chord*; when the chord passes through the focus, it is called the *focal chord*.

140. *To find the equation to a straight line which touches the ellipse in a given point.*

Let x', y' be the co-ordinates of the given point, and x'', y'' those of any other point in the ellipse, near the first.

Then the equation to the line drawn through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x') \dots \dots \dots (1).$$

But these two points being in the ellipse, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2, \text{ and}$$

$$a^2 y''^2 + b^2 x''^2 = a^2 b^2;$$

$$\therefore a^2 (y''^2 - y'^2) + b^2 (x''^2 - x'^2) = 0,$$

$$\therefore \frac{y''^2 - y'^2}{x''^2 - x'^2} = -\frac{b^2}{a^2};$$

$$\therefore \frac{(y'' + y')(y'' - y')}{(x'' + x')(x'' - x')} = -\frac{b^2}{a^2};$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'};$$

therefore equation (1) becomes, by substitution,

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'} (x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y') , then $x'' = x', y'' = y'$, and the secant becomes a tangent at the point (x', y') ; hence the equation to the tangent is

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x'),$$

in which x, y are the variable co-ordinates of any point whatever in the tangent.

141. **COR.** This equation may be presented under a more convenient form; for multiplying each side by $a^2 y'$, we have

$$a^2 y y' - a^2 y'^2 = -b^2 x x' + b^2 x'^2,$$

therefore transposing,

$$a^2 y y' + b^2 x x' = a^2 y'^2 + b^2 x'^2;$$

$$\text{but } a^2 y'^2 + b^2 x'^2 = a^2 b^2, \quad (131),$$

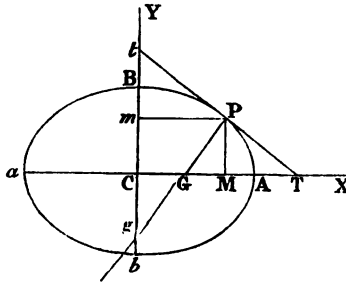
$$\therefore a^2 y y' + b^2 x x' = a^2 b^2,$$

which is the equation most frequently employed.

142. *To find the intersection of the tangent with the axes of x and y .*

The equation to the tangent being

$$a^2 y y' + b^2 x x' = a^2 b^2,$$



- (1) Let it cut the axis of x , as at T ;

$$\text{then } y = 0; \therefore b^2 x x' = a^2 b^2; \therefore x = \frac{a^2}{x'},$$

$$\text{or } CT = \frac{CA^2}{CM}.$$

- (2) Let the tangent cut the axis of y , as at t ;

$$\text{then } x = 0; \therefore a^2 y y' = a^2 b^2; \therefore y = \frac{b^2}{y'},$$

$$\text{or } Ct = \frac{CB^2}{Cm}.$$

Whence it follows, that *each of the semi-axes is a mean proportional between the abscissa of any point, and the part of the axis, intercepted between its intersection with the tangent and the centre.*

$$143. \text{ COR. 1. Since } CT = \frac{a^2}{x'};$$

$$\therefore MT = CT - CM$$

$$= \frac{a^2}{x'} - x'$$

$$= \frac{a^2 - x'^2}{x'}.$$

DEF. The straight line MT , intercepted between the foot of the ordinate, and the point where the tangent meets the axis, is called the *subtangent*.

144. COR. 2. The value of the subtangent being independent of the ordinate y' , it will remain the same for all ellipses described upon the same major axis Aa ; now the circle is a species of ellipse (133); hence, if on the major axis a circle be described, and the ordinate MP be produced upwards to meet the circumference in Q , the tangents applied at P and Q will intersect the axis AX in the same point T .

DEF. The straight line which is drawn from the point of contact at right angles to the tangent is called the *normal*.

145. To find the equation to the normal.

Let TP touch the ellipse in P , and from this point draw the indefinite line Pg at right angles to PT , meeting CA in G , and Cb in g .

Then, because Pg is drawn through the point (x', y') , at right angles to PT , whose equation is

$$y - y' = -\frac{b^2}{a^2} \frac{x'}{y'} (x - x'),$$

its equation will be

$$y - y' = \frac{a^2}{b^2} \frac{y'}{x'} (x - x'),$$

in which x, y are the variable co-ordinates of any point whatever in the line Pg , considered as indefinite.

146. *To find the intersection of the normal with the axes of x and y .*

The equation to the normal being

$$y - y' = \frac{a^2}{b^2} \frac{y'}{x'} (x - x'),$$

first let it cut the axis of x , as at G ; then $y = 0$,

$$\text{and } -y' = \frac{a^2}{b^2} \frac{y'}{x'} (x - x');$$

$$\therefore x - x' = -\frac{b^2}{a^2} x';$$

$$\therefore x' - x = \frac{b^2}{a^2} x',$$

or $CM - CG$, that is, $MG = \frac{b^2}{a^2} x'$.

Next, let the normal cut the axis of y , as at g ;

then $x = 0$,

$$\begin{aligned}\therefore y - y' &= -\frac{a^2}{b^2} \frac{y'}{x'} x' \\ &= -\frac{a^2}{b^2} y'; \\ \therefore y \text{ or } Cg &= -\frac{a^2 - b^2}{a^2} y' .\end{aligned}$$

The negative sign implying that the point g lies *below* the axis AX .

DEF. The line MG , intercepted between the foot of the ordinate, and the point where the normal cuts the axis of x , is called the *subnormal*.

The term *normal* is usually restricted to the finite straight line PG . See Art. 93.

147. To draw a tangent to an ellipse from a given point (x'', y'') without it.

Let x', y' be the unknown co-ordinates of the point of contact.

Then the equation to the tangent being, in general,

$$a^2 yy' + b^2 xx' = a^2 b^2,$$

and the point (x'', y'') being, by hypothesis, a point in the tangent, we have

$$a^2 y'' y' + b^2 x'' x' = a^2 b^2 \dots \dots (1);$$

also the point of contact (x', y') being in the ellipse,

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \dots \dots (2);$$

hence, by means of these two equations, the co-ordinates x', y' of the point of contact may be determined.

Since the equation which results from elimination between (1) and (2) is of the second degree, it follows that there are two

points of contact; in other words, *that two tangents may be drawn to an ellipse, from a given point without it.*

Instead of going through the operation of eliminating, we may, as in Arts. 66 and 94, find the position of the points of contact, by constructing the loci of (1) and (2), in which x' , y' are the variable quantities.

Now the locus of (2) is the given ellipse, and the locus of (1), which is an equation of the first degree, is a straight line, whose position is determined by making x' and y' successively = 0.

If, therefore, in the equation

$$a^2 y' y'' + b^2 x' x'' = a^2 b^2,$$

$$x' = 0, \text{ then } y' = \frac{b^2}{y''},$$

$$y' = 0, \text{ then } x' = \frac{a^2}{x''}.$$

Hence,* take $CM = \frac{a^2}{x''}$ and $CN = \frac{b^2}{y''}$, join M , N , and let MN meet the ellipse in P and p ; these will be the points of contact required.

148. COR. 1. Since the straight line MN , which has just been drawn, determines, by its intersection with the ellipse, the points of contact, it follows that the equation

$$a^2 y'' y' + b^2 x'' x' = a^2 b^2,$$

in which x' and y' are the variables, is the equation to the indefinite straight line joining the points of contact.

149. COR. 2. Because CM is independent of y'' , it will remain the same for all points whose abscissas are = x'' , that is,

* See Fig. p. 51, which may easily be adapted to the ellipse.

for all the points in the indefinite straight line Qq , drawn through Q parallel to CY . Hence the following theorem :

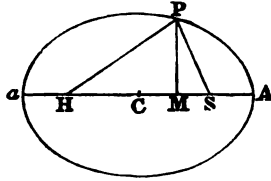
If from the several points of a straight line, perpendicular to the axis CX , pairs of tangents be drawn to the ellipse, the chords joining the points of contact, in each case, will all pass through the same point.

CHAPTER II.

ON THE ELLIPSE REFERRED TO THE FOCUS.

150. *To find the distance of any point in the ellipse from either focus.*

Let S, H be the foci, P any point in the ellipse; to find the value of SP or HP .



(1) Of SP .

In general, the distance between any two points (x, y) and (x', y') is (36)

$$= \sqrt{\{(x - x')^2 + (y - y')^2\}},$$

but the co-ordinates of S , since it is a point in the axis of x , are

$$x' = ae, \quad y' = 0;$$

$$\therefore SP^2 = (x - ae)^2 + y^2$$

$$= (x - ae)^2 + (1 - e^2)(a^2 - x^2) \quad (\text{Arts. 129 and 130.})$$

$$= x^2 - 2aex + a^2e^2 + a^2 - x^2 - e^2a^2 + e^2x^2$$

$$= a^2 - 2aex + e^2x^2$$

$$= (a - ex)^2;$$

$$\therefore SP = a - ex.$$

(2) In like manner it may be proved, that

$$HP = a + ex.$$

DEF. The distance of any point from the focus is called the *focal distance*.

151. COR. Hence, by addition,

$$SP + HP = 2a = Aa;$$

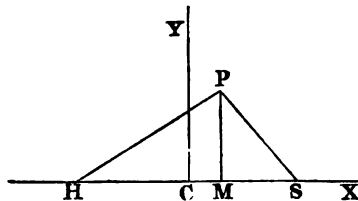
that is, *the sum of the focal distances is equal to the major axis*.

From this property the equation to the ellipse may be deduced, as in the following Article:

152. *To find the locus of a point, whose distances from two fixed points are together always equal to a given quantity $2a$.*

Let S, H be the two fixed points, P the point whose locus is required.

Join S, H , bisect SH in C , let fall the perpendicular PM on HS , which produce indefinitely towards X ; from C draw



CY at right angles to CX , and assume CX and CY as the axes of co-ordinates.

Let $CM = x$, $MP = y$, and $SC = c$.

Then $SP^2 = y^2 + (c - x)^2$
and $HP^2 = y^2 + (c + x)^2$ (1);

$$\therefore HP^2 - SP^2 = (c + x)^2 - (c - x)^2,$$

$$\text{or } (HP + SP)(HP - SP) = 4cx;$$

$$\begin{aligned}\therefore HP - SP &= \frac{4cx}{2a} \\ &= \frac{2cx}{a},\end{aligned}$$

$$\text{but } HP + SP = 2a;$$

therefore adding and subtracting successively,

$$\therefore HP = a + \frac{cx}{a},$$

$$\text{and } SP = a - \frac{cx}{a}.$$

Squaring these values, and adding the result,

$$SP^2 + HP^2 = 2 \left(a^2 + \frac{c^2 x^2}{a^2} \right),$$

$$\text{and also } = 2(y^2 + c^2 + x^2) \text{ from (1);}$$

$$\therefore y^2 + c^2 + x^2 = a^2 + \frac{c^2 x^2}{a^2};$$

$$\therefore y^2 = a^2 - c^2 + \frac{c^2 x^2}{a^2} - x^2$$

$$= a^2 - c^2 - \frac{a^2 - c^2}{a^2} x^2$$

$$= \frac{a^2 - c^2}{a^2} (a^2 - x^2),$$

which is the equation to an ellipse, whose major axis $= 2a$,
and whose minor axis $= 2\sqrt{a^2 - c^2}$.

If $x = 0$, then $y^2 = a^2 - c^2 = b^2$, if b denote the ordinate drawn from C .

153. To find the polar equation to the ellipse, the focus being the pole.

(1) Let S be the pole.

Let $SP = r$, angle $PSA = \omega$;

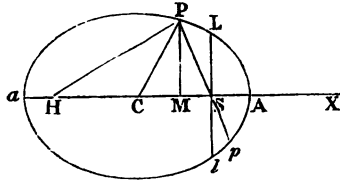
then $r = a - ex$, (Art. 150),

but $x = CS - SM$

$$= ae - r \cos (\pi - \omega)$$

$$= ae + r \cos \omega ;$$

$$\therefore r = a - ae^2 - er \cos \omega ;$$



$$\therefore r(1 + e \cos \omega) = a(1 - e^2) ;$$

$$\therefore r = \frac{a(1 - e^2)}{1 + e \cos \omega} ,$$

which is the equation required.

(2) Let H be the pole.

Let $HP = r'$, and $PHA = \omega'$;

then $r' = a + ex$, (150),

but $x = CM = HM - HC$

$$= r' \cos \omega' - ae ;$$

$$\therefore r' = a + er' \cos \omega' - ae^2 ;$$

$$\therefore r'(1 - e \cos \omega') = a(1 - e^2) ;$$

$$\therefore r' = \frac{a(1 - e^2)}{1 + e \cos \omega'},$$

which is the equation required.

Observation. The same equation may be obtained more concisely thus ;

In the triangle PSH

$$HP^2 = SP^2 + SH^2 - 2SP \cdot SH \cos PSH;$$

therefore substituting for these lines, and observing that

$$\cos PSH = \cos (\pi - \omega) = -\cos \omega,$$

we have

$$(2a - r)^2 = r^2 + 4a^2e^2 + 4aer \cos \omega,$$

$$\text{whence } r = \frac{a(1 - e^2)}{1 + e \cos \omega} \text{ as before.}$$

154. COR. 1. If PS be produced to meet the ellipse in p , then since the angle $ASp = \pi - \omega$, we have

$$Sp = \frac{a(1 - e^2)}{1 - e \cos \omega}.$$

155. COR. 2. Hence

$$\begin{aligned} \frac{1}{SP} + \frac{1}{Sp} &= \frac{1 + e \cos \omega}{a(1 - e^2)} + \frac{1 - e \cos \omega}{a(1 - e^2)} \\ &= \frac{2}{a(1 - e^2)} = \frac{2}{SL}; \end{aligned}$$

that is, the principal semi-parameter is an harmonic mean between the segments of any focal chord.

156. COR. 3. Since $\frac{1}{SP} + \frac{1}{Sp} = \frac{SP + Sp}{SP \cdot Sp}$,

$$\text{and also} = \frac{2}{a(1 - e^2)},$$

$$\therefore SP \cdot Sp = \frac{a}{2} (1 - e^2) (SP + Sp.)$$

157. To find the polar equation to the ellipse, the centre being the pole.

Let $CP = \rho$ and the angle $PCA = \nu$. Fig. p. 107.

Then $\rho^2 = x^2 + y^2$

$$= x^2 + (1 - e^2)(a^2 - x^2), \quad (\text{Art. 129. and 130.})$$

$$= e^2 x^2 + a^2 (1 - e^2)$$

$$= e^2 \rho^2 \cos^2 \nu + a^2 (1 - e^2);$$

$$\therefore \rho^2 (1 - e^2 \cos^2 \nu) = a^2 (1 - e^2);$$

$$\therefore \rho = a \sqrt{\frac{1 - e^2}{1 - e^2 \cos^2 \nu}},$$

which is the equation required.

Observation. Since $b = a\sqrt{1 - e^2}$, (129),

$$\rho = \frac{b}{\sqrt{1 - e^2 \cos^2 \nu}} = b (1 - e^2 \cos^2 \nu)^{-\frac{1}{2}}.$$

Now when the *eccentricity* (135) is very small, the powers of e above the second may, without any sensible error, be neglected.

Hence expanding the above expression,

$$\rho = b (1 + \frac{1}{2} e^2 \cos^2 \nu) \text{ very nearly;}$$

$$\therefore \rho - b = \frac{1}{2} b e^2 \cos^2 \nu \dots \dots \dots (1).$$

Again, since $b = a(1 - e^2)^{\frac{1}{2}} = a(1 - \frac{1}{2}e^2)$,
very nearly,

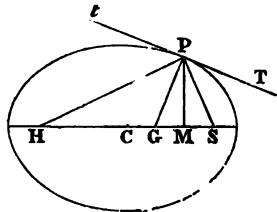
$$\begin{aligned}\rho &= a(1 - \frac{1}{2}e^2)(1 + \frac{1}{2}e^2 \cos^2 \nu) \\ &= a + \frac{1}{2}ae^2 \cos^2 \nu - \frac{1}{2}ae^2 \\ &= a - \frac{1}{2}ae^2(1 - \cos^2 \nu) \\ &= a - \frac{1}{2}ae^2 \sin^2 \nu, \text{ very nearly;} \\ \therefore a - \rho &= \frac{1}{2}ae^2 \sin^2 \nu \dots \dots \dots (2).\end{aligned}$$

It appears therefore from (1) and (2), that *in ellipses of small eccentricity, the increment of the radius vector, in moving from B towards A, is very nearly proportional to $\cos^2$ PCA; and that its decrement, in moving from A towards B, is very nearly proportional to $\sin^2$ PCA.*

158. *To prove that the focal distances of any point make equal angles with the tangent at that point.*

Let TPt be a tangent at the point $P(x', y')$, draw the normal PG , and join S, P and H, P .

$$\text{Then } CG = \frac{a^2 - b^2}{a^2} x' = e^2 x', \quad (146),$$



$$\begin{aligned}\therefore \frac{SG}{HG} &= \frac{SC - CG}{SC + CG} = \frac{ae - e^2 x'}{ae + e^2 x'} = \frac{a - ex'}{a + ex'} \\ &= \frac{SP}{HP} \quad (150); \end{aligned}$$

therefore the angle SPG = the angle HPG , (Euc. vi. 3.)

But $GPT = GPt$, each being a right angle ; $\therefore SPT = HPt$, as was to be proved.

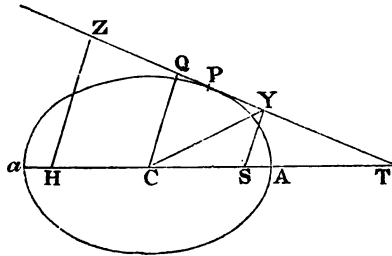
This theorem admits of a more rigorously analytical proof: for the equation to PT being known, and the equations to SP and HP being easily found, we can determine by means of Art. 31 the values of $\tan SPT$, and $\tan HPT$. It will thence be found that

$$\tan SPT = -\tan HPT = \therefore \tan HPt.$$

159. *To find the locus of the points, in which a perpendicular from the focus upon the tangent at any point intersects the tangent.*

Let PT be a tangent at any point $P (x', y')$, and SY a perpendicular let fall from S on PT , meeting it in Y ; to find the locus of Y .

From C let fall the perpendicular CQ on the tangent, and join S, P, H, P and C, Y .



$$\text{Then } CY^2 = CQ^2 + QY^2 \dots\dots (1).$$

But since $SC = CH$, it is plain that

$$YQ = QZ,$$

$$\text{and that } CQ = \frac{1}{2} (SY + HZ).$$

Now if ϕ denote the angle which the focal distances make with the tangent,

$$SY = SP \sin \phi$$

$$\text{and } HZ = HP \sin \phi;$$

$$\therefore CQ = \frac{1}{2} (SP + HP) \sin \phi = a \sin \phi.$$

$$\begin{aligned}
 \text{Also } QY &= \frac{1}{2} YZ = \frac{1}{2} (PY + PZ) \\
 &= \frac{1}{2} (HP + SP) \cos \phi \\
 &= a \cos \phi;
 \end{aligned}$$

$\therefore$ substituting for CQ^2 and QY^2 in (1),

$$\begin{aligned}
 CY^2 &= a^2 (\sin^2 \phi + \cos^2 \phi) \\
 &= a^2;
 \end{aligned}$$

$$\therefore CY = \pm a,$$

whence the locus of Y is a circle, described on the major axis.

160. *The rectangle contained by the perpendiculars let fall from the foci upon the tangent at any point is equal to the square of the semi-axis minor.*

Conceive YS to meet the circumference again in Y' , and

$$\text{Let } SY = r, \quad HZ = r', \quad YSA = \theta.$$

Then if S be the pole, the polar equation to the circle, which is the locus of Y , will be (52)

$$r^2 + 2aer \cos \theta - a^2 (1 - e^2) = 0 \dots \dots \dots (1).$$

In like manner, if H be the pole,

$$r'^2 - 2aer' \cos \theta - a^2 (1 - e^2) = 0 \dots \dots \dots (2).$$

In equation (1) let $\cos \theta$ be changed into $-\cos \theta$, then we shall have, for determining SY' ,

$$r^2 - 2aer \cos \theta - a^2 (1 - e^2) = 0,$$

which is identical with equation (2);

$$\therefore SY' = HZ,$$

but by the theory of equations,

$$SY \cdot SY' = a^2 (1 - e^2);$$

$$\therefore SY \cdot HZ = b^2,$$

as was to be proved.

Observation. This theorem admits, also, of another proof, more strictly analytical than that just given. As the investigation is operose, it will be sufficient to point out the principal steps.

The equation to a tangent at $P(x', y')$ being

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x'),$$

the value of SY , which is drawn through the point $S(ae, 0)$ perpendicular to the tangent, is, by Art. 41,

$$= -b^2 \cdot \frac{a^2 - aex'}{\sqrt{a^4 y'^2 + b^4 x'^2}};$$

therefore, reducing the denominator by means of the equation to the ellipse,

$$SY = -b \sqrt{\frac{a - ex'}{a + ex'}}.$$

In like manner,

$$HZ = -b \sqrt{\frac{a + ex'}{a - ex'}};$$

$$\therefore SY \cdot HZ = b^2,$$

as before.

161. COR. It appears that $SY^2 = BC^2 \cdot \frac{SP}{HP}$;

$\therefore$ if $SY = p$, $SP = r$,

$$p^2 = b^2 \cdot \frac{r}{2a - r}.$$

CHAPTER III.

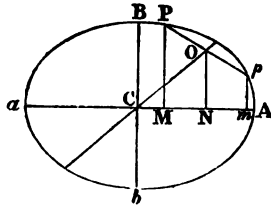
ON THE ELLIPSE REFERRED TO ANY SYSTEM OF CONJUGATE DIAMETERS.

SECTION I.

ON CONJUGATE DIAMETERS IN GENERAL.

162. *To find the locus of the middle points of any number of parallel chords.*

Let Pp be any chord, O its middle point, and X, Y its co-ordinates.



From the points O, P, p let fall the perpendiculars ON, PM, pm on the axis Aa ; then if the equation to Pp be

$$y = ax + \beta,$$

the equation containing the values of y at the points P, p will be

$$y^2 - \frac{2b^2\beta}{a^2a^2 + b^2}y + \frac{b^2(\beta^2 - a^2a^2)}{a^2a^2 + b^2} = 0. \quad (\text{Art. 139.})$$

Now, since in any quadratic equation the coefficient of the second term, with its proper sign, is equal to the sum of the roots with their signs changed,

$$\frac{2b^2\beta}{a^2\alpha^2 + b^2} = PM + pm.$$

But O being the middle point of Pp ,

$$ON = \frac{PM + pm}{2};$$

$$\therefore Y = \frac{b^2\beta}{a^2\alpha^2 + b^2} \dots\dots\dots (1).$$

$$\text{Now } X = \frac{1}{\alpha} (Y - \beta)$$

$$= \therefore - \frac{a^2\alpha\beta}{a^2\alpha^2 + b^2} \dots\dots\dots (2).$$

To obtain the relation between X and Y , we must eliminate β between (1) and (2);

$$\therefore \frac{a^2\alpha^2 + b^2}{b^2} Y = - \frac{a^2\alpha^2 + b^2}{a^2\alpha} X;$$

$$\therefore Y = - \frac{b^2}{a^2\alpha} X.$$

Now, since (α) remains the same for all chords parallel to Pp , (25), the equation just found expresses the relation between the co-ordinates of their middle points; and being of the first degree, the locus required is a *straight line*.

DEF. The straight line, which has been proved to be the locus of the middle points of any number of parallel chords, is called a *diameter*, and the points in which it intersects the curve are called the *vertices*.

163. COR. The equation $Y = - \frac{b^2}{a^2\alpha} X$, is the equation

to a straight line passing through the origin, which in this case is the centre; hence *every diameter must pass through the centre.*

164. *A diameter being drawn through a given point, to find the equation to any one of its ordinates.*

If x', y' be the co-ordinates of the given point, the equation to the diameter drawn through it will be

$$y = \frac{y'}{x'} x \dots\dots\dots (1).$$

$$\text{Let } y = ax + \beta \dots\dots\dots (2),$$

be the required equation to any ordinate,

$$\text{then } \frac{y'}{x'} = -\frac{b^2}{a^2 a} \quad (162);$$

$$\therefore a = -\frac{b^2}{a^2} \cdot \frac{x'}{y'},$$

hence, any ordinate to a diameter passing through (x', y') has for its equation

$$y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} x + \beta.$$

165. *Cor.* Comparing this equation with the equation to the tangent (Art. 140), it appears, that *the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.*

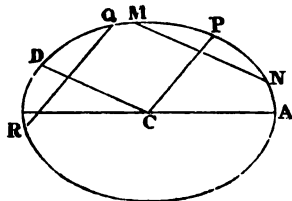
166. *Any two diameters being given, if the ordinates of one be parallel to the other, the ordinates of the latter will be parallel to the former.*

$$\text{Let } y = ax \dots\dots\dots (1),$$

$$y = a'x \dots\dots\dots (2),$$

be the equations to any two diameters CP , CD , then the

equations to any ordinates MN , QR , to the first and second



respectively, will be (164)

$$y = -\frac{b^2}{a^2 a} x + \beta \dots\dots\dots (1'),$$

$$y = -\frac{b^2}{a'^2 a'} x + \beta' \dots\dots\dots (2').$$

Let the ordinate MN be now supposed parallel to the diameter CD , then (25)

$$-\frac{b^2}{a'^2 a} = a',$$

$$\text{or } a = -\frac{b^2}{a'^2 a'};$$

therefore the equation to QR becomes by substitution in (2'),

$$y = ax + \beta',$$

that is, QR the second ordinate is parallel to the first diameter CP , which was to be proved.

Wherefore each of these diameters is parallel to the ordinates of the other.

Diameters thus related are said to be *conjugate* to each other.

167. COR. 1. Hence, when the two diameters

$$y = ax,$$

$$y = a'x,$$

are conjugate to each other,

$$aa' = -\frac{b^2}{a^2}.$$

168. COR. 2. Therefore if

$$y = ax$$

be any diameter,

$$y = -\frac{b^2}{a^2 a} x$$

will be the diameter conjugate to it. The number of pairs of conjugate diameters is therefore unlimited.

If $a = 0$, or the first diameter be Aa , fig. p. 115, then

$$y = -\frac{b^2}{a^2 0} x = \infty \cdot x,$$

therefore the diameter conjugate to Aa , being at right angles to it, is Bb ; that is, *the axes of the ellipse are conjugate diameters.*

169. COR. 3. If (x', y') be any point in the ellipse, the diameter passing through it is

$$y = \frac{y'}{x'} x,$$

$$\therefore y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} x$$

is the corresponding conjugate diameter.

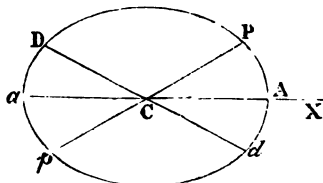
But the equation to a tangent drawn through (x', y') is (140),

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x') \dots \dots \dots (1),$$

whence it follows, *that the tangent applied to the vertex of any diameter is parallel to the corresponding conjugate diameter.*

170. It has just been shown, in Cor. 2, that the axes of the ellipse are conjugate diameters; we shall now prove, *that the axes are the only pair of conjugate diameters which can be at right angles to each other.*

For, if possible, let CP , CD be a pair of rectangular



conjugate diameters, different from the axes, and let

angle $PCA = \theta$, angle $DCA = \theta'$.

Now $\theta' = DCA = DCP + PCA$

$$= \frac{\pi}{2} + \theta, \text{ by hypothesis;}$$

$$\text{but } -\frac{b^2}{a^2} = aa' \text{ (167)} = \therefore \tan \theta \tan \theta'$$

$$= \frac{\sin \theta}{\cos \theta} \cdot \frac{\sin \theta'}{\cos \theta'},$$

$$\therefore a^2 \sin \theta \sin \theta' + b^2 \cos \theta \cos \theta' = 0 \dots \dots \dots (1),$$

$$\text{but } \sin \theta' = \sin \left(\frac{\pi}{2} + \theta \right) = \sin \left(\frac{\pi}{2} - \theta \right) = \cos \theta,$$

$$\text{and } \cos \theta' = \cos \left(\frac{\pi}{2} + \theta \right) = -\cos \left(\frac{\pi}{2} - \theta \right) = -\sin \theta;$$

therefore by substitution in (1),

$$(a^2 - b^2) \sin \theta \cos \theta = 0,$$

$$\text{or } \frac{1}{2} (a^2 - b^2) \sin 2\theta = 0.$$

Now, since a is $> b$, this equation can be satisfied only by supposing

$$\sin 2\theta = 0, \therefore 2\theta = 0, \text{ or } 2\theta = \pi;$$

$$\therefore \theta = 0, \quad \text{or} \quad = \frac{\pi}{2};$$

$$\text{and } \therefore \theta' = \frac{\pi}{2}, \quad \text{or} \quad = 0;$$

whence, CP and CD must coincide with CA and CB respectively, and the axes, therefore, are the only conjugate diameters which can be at right angles to each other.

171. *To find the equation to the ellipse, when it is referred to any two conjugate diameters, as axes.*

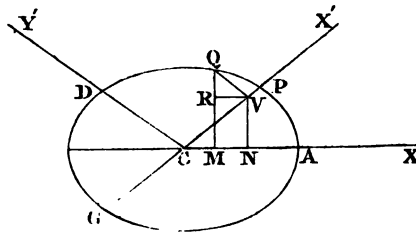
Let CP , CD , any system of conjugate diameters, be assumed as axes, take Q any point in the ellipse, and draw QM perpendicular to CX , and QV parallel to CY' , and from V draw VN , VR , parallel to QM , CA , respectively.

Assume $CM = x$, $MQ = y$; $CV = x'$, $VQ = y'$,

also let $CP = a'$, $CD = b'$, angle $PCA = \theta$, and angle $DCA = \phi$.

The object now is, to determine the relation between x' and y' .

In general $a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2 \dots \dots \dots (1)$.



$$\text{But } x = CN - MN = x' \cos \theta + y' \cos \phi,$$

$$y = MR + RQ = x' \sin \theta + y' \sin \phi^*;$$

* These are the formulas for passing from rectangular to oblique co-ordinates. See Art. 76.

therefore by substitution in (1),

$$a^2 (x' \sin \theta + y' \sin \phi)^2 + b^2 (x' \cos \theta + y' \cos \phi)^2 = a^2 b^2,$$

or developing, and arranging the result,

$$(a^2 \sin^2 \phi + b^2 \cos^2 \phi) y'^2 + (a^2 \sin^2 \theta + b^2 \cos^2 \theta) x'^2 \\ + 2 (a^2 \sin \theta \sin \phi + b^2 \cos \theta \cos \phi) x' y' = a^2 b^2.$$

But it was shown in Art. 167, that

$$a \alpha', \text{ or } \tan \theta \cdot \tan \phi = -\frac{b^2}{a^2},$$

$$\therefore a^2 \sin \theta \sin \phi + b^2 \cos \theta \cos \phi = 0;$$

therefore the term involving $x' y'$ vanishes, and the equation required is of the form

$$(a^2 \sin^2 \phi + b^2 \cos^2 \phi) y'^2 + (a^2 \sin^2 \theta + b^2 \cos^2 \theta) x'^2 = a^2 b^2 \dots (2).$$

Now when the axis CX' meets the ellipse, $y' = 0$, and $x' = CP = a'$;

$$\therefore a^2 \sin^2 \theta + b^2 \cos^2 \theta = \frac{a^2 b^2}{a'^2};$$

also when the axis CY' meets the ellipse, $x' = 0$, and $y' = CD = b'$;

$$\therefore a^2 \sin^2 \phi + b^2 \cos^2 \phi = \frac{a^2 b^2}{b'^2};$$

therefore by substitution in (2),

$$\frac{a^2 b^2}{b'^2} y'^2 + \frac{a^2 b^2}{a'^2} x'^2 = a^2 b^2, \text{ or dividing each term by } a^2 b^2,$$

$$\frac{y'^2}{b'^2} + \frac{x'^2}{a'^2} = 1,$$

$$\text{or, } a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2,$$

either of which is the equation required.

172. COR. 1. Hence, suppressing the accents of the co-ordinates,

$$y = \frac{b'}{a'} \sqrt{a'^2 - x^2} \dots\dots\dots (1).$$

173. COR. 2. To find the form of the equation, when the co-ordinates originate at P , the vertex of the diameter CP .

Let $PV = x'$, then

$$x = CP - PV = a' - x'.$$

Substituting this value of x in Cor. 1, we have

$$y = \pm \frac{b'}{a'} \sqrt{2a'x' - x'^2},$$

or suppressing the accent of x ,

$$y = \pm \frac{b'}{a'} \sqrt{2a'x - x^2} \dots\dots\dots (2),$$

which is the equation required.

174. COR. 3. The equations (1) and (2), are of the same form as the equations in terms of the axes, and express, when translated into geometrical language, a property of which that in Art. 132 is only a particular case.

$$\text{For } a'^2 - x^2 = (a' + x)(a' - x) = PV \cdot VG,$$

$$\text{and } 2a'x - x^2 = (2a' - x)x = PV \cdot VG;$$

$$\therefore VQ^2 = \frac{CD^2}{PC^2} PV \cdot VG,$$

$$\text{or } PV \cdot VG : QV^2 :: PC^2 : CD^2;$$

that is, *the rectangle contained by the segments of any diameter is to the square of the ordinate, as the square of the semi-diameter is to the square of its semi-conjugate.*

175. It appears from the preceding proposition, that the equation to the ellipse, whether the axes be rectangular or oblique, is always of the same form; whence it follows, that, when the ellipse is referred to any two conjugate diameters,

(1) If the equation to the major axis Aa , be $y = ax$, then the equation to the minor axis Bb will be

$$y = -\frac{b'^2}{a'^2}x,$$

and

(2) The equation to the tangent will be

$$a'^2yy' + b'^2xx' = a'^2b'^2.$$

176. *To find the intersection of the tangent with any two conjugate diameters, considered as axes.*

Let a tangent applied at any point Q^* , be conceived to meet CP in T and CD in t , and draw the ordinates QV , Qv , to the diameters CP , CD respectively; the equation to the tangent being

$$a'^2yy' + b'^2xx' = a'^2b'^2,$$

let the tangent meet CX' as at T , then $y = 0$;

$$\therefore x = \frac{a'^2}{x'}, \text{ or } CT = \frac{CP^2}{CV}.$$

Let the tangent meet CY' as at t , then $x = 0$,

$$\therefore y = \frac{b'^2}{y'}, \text{ or } Ct = \frac{CD^2}{Cv};$$

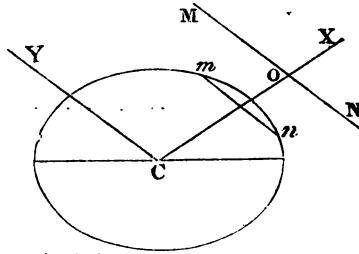
therefore the points of intersection required are found.

Hence, *each semi-diameter is a mean proportional between the abscissa, and the part of the diameter cut off by the tangent.* See Art. 142.

* See Fig. p. 121.

177. *If from the several points of a straight line given in position pairs of tangents be drawn to an ellipse, the straight lines, which join the corresponding points of contact, will all pass through the same point.*

Let C be the centre of the ellipse, MN the given straight line.



Draw any chord mn parallel to MN , and bisect it by the diameter CX ; from C draw CY parallel to mn or MN , then CX , CY are conjugate diameters (166); and if the ellipse be referred to these as axes, its equation will be (171)

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \dots\dots\dots (1).$$

From any point (x'', y'') in MN , let a pair of tangents be drawn to the ellipse, then it may easily be shewn, as in Art. 148, that the equation to the straight line joining the points of contact is

$$a'^2 y'' y' + b'^2 x'' x' = a'^2 b'^2 \dots\dots\dots (2),$$

in which x' , y' are the variable co-ordinates of the point of contact.

Let the line (2) cut the axis of x , then $y' = 0$, and

$$\therefore x' = \frac{a'^2}{x''};$$

hence the point of intersection will be the same for all points whose abscissa = x'' , that is, for all points in the straight line MN , as was to be proved.

178. **COR.** The point of intersection is situated on the diameter, conjugate to that which is parallel to the given line.

179. *If from the point of intersection of two tangents a diameter be drawn, it will bisect the straight line joining the points of contact.*

For the equation to an ordinate to the diameter passing through (x'', y'') , is (164)

$$y = -\frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x + \beta \dots \dots \dots (1),$$

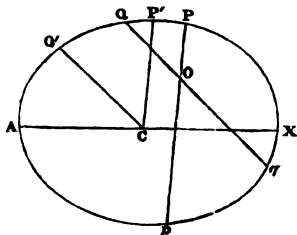
and the equation to the straight line, which joins the points of contact, is

$$y' = -\frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x - \frac{b'^2}{y''} \dots \dots \dots (2);$$

hence the latter, being parallel to the former, is also an ordinate, and is therefore bisected.

180. *If through any point within or without an ellipse two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will bear a constant ratio to the rectangle contained by the segments of the other.*

Let O be any point *within* the ellipse, and through O draw the two straight lines Pp and Qq , whose position is



supposed known, meeting the ellipse in the points P, p and Q, q ; to prove that

$$OP \cdot Op : OQ \cdot Oq \text{ in a constant ratio.}$$

From O (α , β) conceive the perpendicular ON let fall upon AX *.

Let $OP = r$, $OQ = r'$, and let θ , θ' denote the angles which r , r' make with AX .

Then referring the curve to rectangular co-ordinates,

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots \dots \dots (1).$$

$$\text{But } y = \beta + r \sin \theta,$$

$$\text{and } x = \alpha + r \cos \theta;$$

therefore by substitution in (1)

$$a^2 \{\beta + r \sin \theta\}^2 + b^2 \{\alpha + r \cos \theta\}^2 = a^2 b^2,$$

or developing, arranging the result, and dividing each term by the coefficient of r^2 ,

$$r^2 + \frac{2\{a^2 \beta \sin \theta + b^2 \alpha \cos \theta\}}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} r + \frac{a^2 \beta^2 + b^2 \alpha^2 - a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta} = 0,$$

in which the values of r are OP , Op ; therefore by the theory of equations,

$$OP \cdot Op = \frac{a^2 \beta^2 + b^2 \alpha^2 - a^2 b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}.$$

In like manner

$$OQ \cdot Oq = \frac{a^2 \beta'^2 + b^2 \alpha'^2 - a^2 b^2}{a^2 \sin^2 \theta' + b^2 \cos^2 \theta'};$$

$$\therefore OP \cdot Op : OQ \cdot Oq :: a^2 \sin^2 \theta' + b^2 \cos^2 \theta' : a^2 \sin^2 \theta + b^2 \cos^2 \theta,$$

and as θ , θ' are known, these rectangles are to each other in a constant ratio, as was to be proved.

181. Cor. 1. From C draw CP' , CQ' parallel to Pp , Qq respectively, then if x' , y' be the co-ordinates of P' ,

$$x'^2 = CP'^2 \cos^2 \theta,$$

$$\text{and } y'^2 = CP'^2 \sin^2 \theta;$$

* Sec Art. 120.

$$\therefore a^2 y'^2 + b^2 x'^2 = (a^2 \sin^2 \theta + b^2 \cos^2 \theta) CP'^2 = \therefore a^2 b^2;$$

$$\therefore a^2 \sin^2 \theta + b^2 \cos^2 \theta = \frac{a^2 b^2}{CP'^2}.$$

In like manner,

$$a^2 \sin^2 \theta' + b^2 \cos^2 \theta' = \frac{a^2 b^2}{CQ'^2},$$

$$\text{wherefore } OP \cdot Op : OQ \cdot Oq :: CP'^2 : CQ'^2.$$

182. COR. 2. Hence also, if the point O be without the ellipse, and the points P, p and Q, q coincide, then OP, OQ will become tangents, and

$$OP^2 : OQ^2 :: CP'^2 : CQ'^2,$$

$$\text{or } OP : OQ :: CP' : CQ'.$$

Hence it may easily be shewn, that if any polygon be drawn, circumscribing an ellipse, the products of its alternate segments are equal.

183. COR. 3. If Pp be supposed to pass through the centre, and to bisect Qq in O , then the semi-diameter which is parallel to OQ will be conjugate to CP , and we shall have

$$PO \cdot Op : OQ^2 :: CP^2 : CD^2,$$

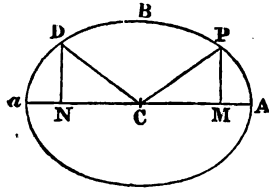
as in Art. 174

SECTION II.

ON THE PROPERTIES OF CONJUGATE DIAMETERS.

184. *A DIAMETER being drawn through a given point (x', y') , to find the co-ordinates of the point in which the diameter conjugate to it meets the ellipse.*

Let P be the given point, and CP , CD any two semi-conjugate diameters, then the equation to CP being



$$y = \frac{y'}{x'} x \dots\dots\dots (1),$$

the equation to CD will be (169)

$$y = -\frac{b^2 x'}{a^2 y'} x \dots\dots\dots (2),$$

therefore the co-ordinates of the point D , in which CD cuts the ellipse, will be determined by combining (2) with the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2 \dots\dots\dots (3).$$

Hence, substituting in (3) the value of y in (2), we have

$$\left\{ a^2 \frac{b^4}{a^4} \cdot \frac{x'^2}{y'^2} + b^2 \right\} x^2 = a^2 b^2,$$

or, dividing by b^2 ,

$$\left(\frac{b^2}{a^2} \frac{x'^2}{y'^2} + 1 \right) x^2 = a^2;$$

$$\therefore (b^2 x'^2 + a^2 y'^2) x^2 = a^4 y'^2;$$

$$\therefore a^2 b^2 x^2 = a^4 y'^2;$$

$$\therefore x^2 = \frac{a^2}{b^2} y'^2,$$

$$\text{or } x = \pm \frac{a}{b} y';$$

$$\text{therefore also, } y = -\frac{b^2}{a^2} \frac{x'}{y'} x = \mp \frac{b}{a} x',$$

the signs of x and y being different, as they ought to be.

These results, when exhibited geometrically, present a well known property of the ellipse.

185. *The sum of the squares of any two semi-conjugate diameters is equal to the sum of the squares of the semi-axes.*

Let CP , CD be any two semi-conjugate diameters, then denoting them by a' , b' respectively,

By the last Article,

$$x^2 + y^2 \text{ or } b'^2 = \frac{a^2}{b^2} y'^2 + \frac{b^2}{a^2} x'^2 \dots\dots\dots (1),$$

$$\text{but } y'^2 = \frac{b^2}{a^2} (a^2 - x'^2) \text{ and } x'^2 = \frac{a^2}{b^2} (b^2 - y'^2);$$

therefore by substitution in (1) we have

$$b'^2 = a^2 - x'^2 + b^2 - y'^2,$$

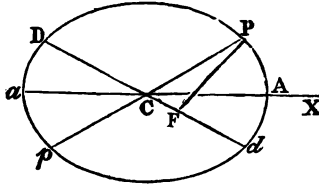
$$\text{also } a'^2 = x'^2 + y'^2;$$

$$\therefore a'^2 + b'^2 = a^2 + b^2$$

as was to be proved.

186. *If at the vertices of any two conjugate diameters tangents be applied, so as to form a parallelogram, the area of all such parallelograms is constant.*

Let Pp , Dd be any two conjugate diameters, drawn through the points (x', y') and (x, y) , and conceive tangents



to be applied at their vertices, so as to form, when produced to meet one another, a parallelogram: from P let fall the perpendicular PF on Dd .

Then the area of the whole parallelogram

$$= 4 CD \cdot PF.$$

Now by Art. (42) PF , the perpendicular from $P (x', y')$, on $CD \left(y = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} x \right)$,

$$= \frac{y' + \frac{b^2}{a^2} \frac{x'^2}{y'^2}}{\sqrt{1 + \frac{b^4}{a^4} \frac{x'^2}{y'^2}}} = \frac{a^2 y'^2 + b^2 x'^2}{\sqrt{a^4 y'^2 + b^4 x'^2}};$$

but $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ and from (1) in Art. 185

$$a^4 y'^2 + b^4 x'^2 = a^2 b^2 CD^2;$$

$\therefore$ by substitution, $CD \cdot PF = ab$;

hence the area of the parallelogram $= 4ab$, and therefore is constant, as was to be proved.

187. COR. 1. Let γ denote the angle PCD , then since $CD = b'$ and $PF = PC \sin PCF = a' \sin \gamma$, we have by substitution in the equation $ab = CD \cdot PF$,

$$ab = a'b' \sin \gamma.$$

188. COR. 2. Hence the value of PF may be found.

$$\text{For } PF = \frac{ab}{CD},$$

$$\text{but } CD^2 = a^2 + b^2 - a'^2 \text{ (185);}$$

$$\therefore PF = \frac{ab}{\sqrt{a^2 + b^2 - a'^2}}.$$

189. To find the magnitude and position of two equal conjugate diameters.

$$\text{In general, } a^2 + b^2 = a'^2 + b'^2, \text{ (185)}$$

$$\text{Let } a' = b';$$

$$\therefore 2a'^2 = a^2 + b^2;$$

$$\therefore a' = \pm \sqrt{\frac{a^2 + b^2}{2}} \dots\dots\dots (1),$$

hence, the *magnitude* of the equal conjugate diameters is found.

Again, their *position* may be determined.

$$\text{For } ab = a'b' \sin \gamma$$

$$= \therefore a'^2 \sin \gamma, \text{ when } a' = b';$$

$$\therefore \sin \gamma = \frac{ab}{a'^2}$$

$$= \frac{2ab}{a^2 + b^2} \dots\dots\dots (2),$$

whence their mutual inclination is known.

Again, since they are equal, they are symmetrically placed with respect to the major axis, and are therefore equally inclined to it; but in general (167),

$$\tan PCA \cdot \tan DCA = -\frac{b^2}{a^2} \text{ (Fig. p. 129),}$$

$$\text{or } \tan PCA \cdot \tan DCa = \frac{b^2}{a^2} = \tan^2 PCA;$$

$$\therefore \tan PCA = \pm \frac{b}{a} \dots\dots\dots (3),$$

wherefore their inclination to the major axis is known.

190. COR. It thence appears that the equal conjugate diameters are parallel to the straight lines BA , Ba .

191. *Of all systems of conjugate diameters those that are equal contain the greatest angle.* (Fig. p. 131.)

$$\text{For, in general, } \sin \gamma = \frac{ab}{a'b'} \text{ (187),}$$

therefore the angle PCd is a *minimum*, or PCD a *maximum*, when the product $a'b'$ is a *maximum*; that is, when $a' = b'$, as was to be proved.

192. COR. Hence it may be shown, that of all systems of conjugate diameters, the sum of those that are rectangular is the least, and of those that are equal, the greatest.

$$\begin{aligned} \text{For } a' + b' &= \sqrt{(a'^2 + b'^2 + 2ab')} \\ &= \sqrt{\left(a^2 + b^2 + \frac{2ab}{\sin \gamma}\right)}: \text{ wherefore} \end{aligned}$$

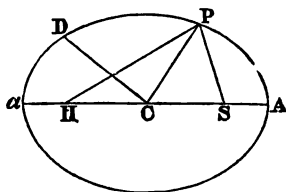
(1) $a' + b'$ is a *maximum*, when $\sin \gamma$ is a *minimum*, that is, when $a' = b'$: and

(2) $a' + b'$ is a *minimum*, when $\sin \gamma$ is a *maximum*, that is, when $\gamma = \frac{\pi}{2}$, or when the conjugate diameters are rectangular.

193. *The rectangle contained by the focal distances of any point is equal to the square of the corresponding semi-conjugate diameter.*

Let P be any point (x, y) , CD the semi-diameter conjugate to CP , join P, S and P, H ; to prove that

$$SP \cdot HP = CD^2.$$



$$\text{In general, } CD^2 = a^2 + b^2 - CP^2 \quad (185)$$

$$= a^2 + b^2 - (x^2 + y^2)$$

$$= a^2 + b^2 - x^2 - (1 - e^2)(a^2 - x^2)$$

$$= a^2 + b^2 - x^2 - a^2 + x^2 + e^2 a^2 - e^2 x^2$$

$$= b^2 + a^2 e^2 - e^2 x^2$$

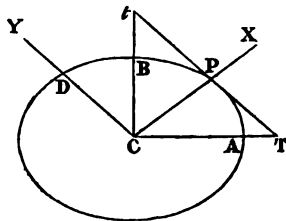
$$= a^2 - e^2 x^2 \dots\dots\dots (1).$$

$$\text{But } a^2 - e^2 x^2 = (a - ex)(a + ex)$$

$$= SP \cdot HP; \quad (150)$$

$$\therefore SP \cdot HP = CD^2.$$

194. *Let CP, CD be any two semi-conjugate diameters,*



and let a tangent at P meet the axes of the ellipse in T and t , to prove that $PT \cdot Pt = CD^2$.

If CP , CD be assumed as the axes of co-ordinates, then the equations to CA , CB are, respectively, (175)

$$y = ax,$$

$$\text{and } y = -\frac{b'^2}{a'^2}x.$$

Let $x = a'$ or CP , then y or $PT = aa'$, in the first,

$$\text{and } y \text{ or } Pt = -\frac{b'^2}{aa'}, \text{ in the second;}$$

$$PT \cdot Pt = -b'^2 = CD^2.$$

The product $PT \cdot Pt$ is negative, because PT , Pt , being situated on opposite sides of the axis AX , have different signs.

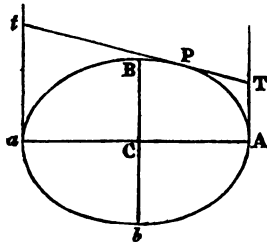
195. COR. It may be proved, in precisely the same manner, that if the tangent at P meet *any two conjugate diameters whatever* in T and t , then

$$PT \cdot Pt = CD^2.$$

196. If a tangent at any point P meet tangents at the vertices of the major axis in T , and t , to prove that

$$AT \cdot at = BC^2.$$

The equation to the tangent at $P(x', y')$, is



$$a^2yy' + b^2xx' = a^2b^2.$$

Let $x = a = CA$,

then $a^2 y y' = b^2 a (a - x')$;

$$\therefore y \text{ or } AT = \frac{b^2}{ay'} (a - x');$$

in like manner, if $x = -a = Ca$,

$$y \text{ or } at = \frac{b^2}{ay'} (a + x');$$

$$\therefore AT \cdot at = \frac{b^2}{a^2 y'^2} (a^2 - x'^2) \dots\dots\dots (1).$$

$$\text{But } y'^2 = \frac{b^2}{a^2} (a^2 - x'^2),$$

$$\text{or } \frac{a^2 - x'^2}{y'^2} = \frac{a^2}{b^2};$$

therefore, by substitution in (1),

$$AT \cdot At = b^2 = BC^2.$$

197. It may in like manner be proved, that if Pp' , $D'd'$ be *any two conjugate diameters whatever*, and the tangent at P meet tangents applied at the vertices of the former, in T and t ,

$$P'T \cdot p't = CD^2.$$

SECTION III.

ON SUPPLEMENTAL CHORDS.

DEF. If from the vertices of any diameter two straight lines be drawn to any point in the ellipse, they are called *Supplemental Chords*.

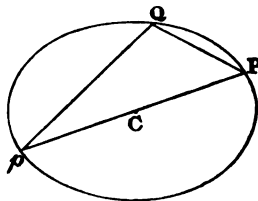
The chords drawn from the vertices of the major axis are called the Principal Supplemental Chords.

198. *Any two supplemental chords being drawn, and the equation to either of them being given; to find the equation to the other.*

The ellipse being referred to any two conjugate diameters, its equation will be

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \dots\dots\dots (1).$$

Through any point $P(x', y')$ draw the diameter Pp , and let PQ, pQ be any two supplemental chords; then if the equation to PQ be



$$y - y' = a(x - x') \dots\dots\dots (2),$$

it is required to find the equation to pQ .

The co-ordinates of P being x', y' , those of p will be $-x', -y'$, therefore the equation to pQ will be (20) of the form

$$y + y' = \alpha' (x + x') \dots\dots\dots (3),$$

in which α' is to be found.

Since the straight lines, whose equations are (2) and (3), intersect in Q , the co-ordinates of that point will be the same in both equations; we have, therefore, by multiplying them together,

$$y^2 - y'^2 = \alpha \alpha' (x^2 - x'^2);$$

$$\therefore \alpha \alpha' = \frac{y^2 - y'^2}{x^2 - x'^2} \dots\dots\dots (4);$$

but x' and y' being the co-ordinates of P , a point in the ellipse, they will satisfy equation (1);

$$\therefore a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2.$$

Subtracting this from (1), we have

$$a'^2 (y^2 - y'^2) + b'^2 (x^2 - x'^2) = 0;$$

$$\therefore \frac{y^2 - y'^2}{x^2 - x'^2} = -\frac{b'^2}{a'^2};$$

therefore by substitution in (4),

$$\alpha \alpha' = -\frac{b'^2}{a'^2}; \quad \therefore \alpha' = -\frac{b'^2}{a'^2 \alpha},$$

and the equation to pQ becomes, by substitution,

$$y + y' = -\frac{b'^2}{a'^2 \alpha} (x + x').$$

199. Cor. Let Pp coincide with the major axis Aa , then the equation to aQ , drawn through the point $(-a, 0)$, will be

$$y = \alpha (x + a),$$

therefore the equation to AQ , drawn through the point $A (+a, 0)$, will be

$$y = -\frac{b'^2}{a'^2 a} (x - a).$$

200. *If two diameters be drawn parallel to any two supplemental chords, they will be conjugate to each other.*

The equations to any two supplemental chords being

$$y - y' = a (x - x') \dots \dots \dots (1),$$

$$\text{and } y + y' = -\frac{b'^2}{a'^2 a} (x + x') \dots \dots \dots (2),$$

let any diameter be drawn parallel to (1), then its equation will be

$$y = ax;$$

therefore the equation to its conjugate being (168)

$$y = -\frac{b'^2}{a'^2 a} x,$$

it follows (25) that the latter is parallel to (2), as was to be proved.

201. *COR. 1. Hence may be drawn a diameter, which shall be conjugate to a given diameter.*

Let Pp be the given diameter, and

(1) Let the major axis of the ellipse be given.

From a draw aR parallel to Pp , meeting the ellipse in R , and join R, A^* ; then if Dd be drawn through C , parallel to RA , it will be conjugate to Pp .

(2) If the major axis be not given.

Draw any diameter whatever Rr ; through r draw rQ

* The Student will have no difficulty in supplying a figure, as the same letters are invariably used to represent the same points and lines.

$$\begin{aligned}\tan A Q a &= \frac{a' - a}{1 + a a'} \quad (31) \\ &= \frac{a' - a}{b^2} \dots\dots (1), \text{ since } a' = -\frac{b^2}{a^2 a}.\end{aligned}$$

$$\text{Now } a' = \tan Q A X = -\tan Q A a = -\frac{y'}{a - x'},$$

$$\text{and } a = \tan Q a X \dots\dots\dots = \frac{y'}{a + x'},$$

$$\begin{aligned}\therefore a' - a &= -y' \cdot \left(\frac{1}{a - x'} + \frac{1}{a + x'} \right) \\ &= -\frac{y'}{a^2 - x'^2} \cdot 2a \\ &= -\frac{2ab^2}{a^2 y'};\end{aligned}$$

therefore by substitution in (1),

$$\tan A Q a = -\frac{2ab^2}{y'(a^2 - b^2)}.$$

204. COR. 1. Since the sign of this quantity is negative, the proposed angle is always obtuse; therefore, also, the angle contained by any two conjugate diameters is obtuse.

205. COR. 2. The angle AQa will be the greatest possible, when y' is so, that is, when $y' = b$, or the point Q coincides with B , the vertex of the minor axis. At this point the principal supplemental chords are equal, and their inclination to the major axis is

$$= \tan^{-1} \frac{b}{a} *.$$

206. COR. 3. Hence, the conjugate diameters which are parallel to these chords are also equal, and contain the greatest possible angle. See Art. 191.

* By this expression is meant the angle whose tangent is $\frac{b}{a}$.

207. To draw two conjugate diameters that shall make a given angle with each other.

The ellipse being referred to its axes, let $2a'$, $2b'$ denote the conjugate diameters required, and γ the supplement of the angle, at which they are inclined to each other.

Then, since (Arts. 185, and 187)

$$a'^2 + b'^2 = a^2 + b^2 \dots\dots\dots (1),$$

$$\text{and } a'b' = \frac{ab}{\sin \gamma} \dots\dots\dots (2),$$

we have, by adding twice the second equation to the first,

$$a'^2 + b'^2 + 2a'b' = a^2 + b^2 + \frac{2ab}{\sin \gamma},$$

therefore, extracting the square root,

$$a' + b' = \pm \sqrt{a^2 + b^2 + \frac{2ab}{\sin \gamma}}.$$

In like manner,

$$a' - b' = \pm \sqrt{a^2 + b^2 - \frac{2ab}{\sin \gamma}},$$

therefore, by addition and subtraction successively,

$$a' = \pm \frac{1}{2} \sqrt{a^2 + b^2 + \frac{2ab}{\sin \gamma}} \pm \frac{1}{2} \sqrt{a^2 + b^2 - \frac{2ab}{\sin \gamma}},$$

$$b' = \pm \frac{1}{2} \sqrt{a^2 + b^2 + \frac{2ab}{\sin \gamma}} \mp \frac{1}{2} \sqrt{a^2 + b^2 - \frac{2ab}{\sin \gamma}};$$

whence the *magnitude* of the required diameters is determined.

Again, since $PCA = DCA - DCP$, Fig. p. 129,

$$\tan PCA = \frac{\tan DCA - \tan DCP}{1 + \tan DCA \tan DCP},$$

or, retaining the notation already used,

$$\alpha = \frac{a' + \tan \gamma}{1 - a' \tan \gamma};$$

$$\text{but } a a' = -\frac{b^2}{a^2}; \quad \therefore a' = -\frac{b^2}{a^2 a},$$

and, by substitution,

$$a = \frac{-\frac{b^2}{a^2 a} + \tan \gamma}{1 + \frac{b^2}{a^2 a} \tan \gamma};$$

$$\therefore a^2 - \frac{b^2}{a^2} \tan \gamma \cdot a = -\frac{b^2}{a^2} - a \tan \gamma,$$

$$\text{or } a^2 - \left(1 - \frac{b^2}{a^2}\right) \tan \gamma \cdot a = -\frac{b^2}{a^2};$$

$$\therefore a = \frac{a^2 - b^2}{2 a^2} \tan \gamma \pm \frac{1}{2 a^2} \sqrt{(a^2 - b^2)^2 \tan^2 \gamma - 4 a^2 b^2},$$

therefore, the *position*, also, of the diameters is determined.

The problem would be impossible, if

$$\tan^2 \gamma \text{ were } < \frac{4 a^2 b^2}{(a^2 - b^2)^2},$$

$$\text{or } \tan \gamma < \frac{2 a b}{a^2 - b^2}.$$

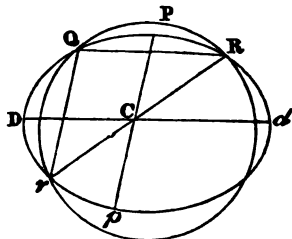
But γ being an acute angle, it will be a *minimum* when the diameters are equal (191), and in that case

$$\tan \gamma = \frac{2 a b}{a^2 - b^2},$$

therefore, as $\tan \gamma$ can never be *less* than $\frac{2 a b}{a^2 - b^2}$, the problem is always possible.

The same problem admits of the following geometrical solution.

Draw any diameter whatever Rr , and upon it describe



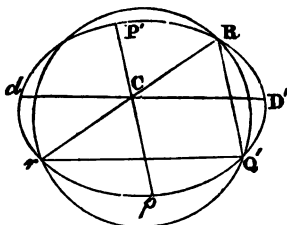
a segment of a circle, containing an angle equal to the given angle, and cutting the ellipse in Q ; draw Qr , QR , and, parallel to these, the diameters Pp , Dd ; these will be the diameters required.

For, being parallel to the supplemental chords QR , Qr , they are conjugate to each other (200), and the angle

$$PCD = \angle RQr,$$

and is therefore equal to the given angle.

The problem admits of a second solution; for the circle



will cut the ellipse again in some point Q' ; draw therefore the supplemental chords $Q'R$, $Q'r'$; then if $P'p'$ and $D'd'$ be drawn through the centre parallel to $Q'R$, $Q'r'$, they will be the diameters required.

For they are evidently conjugate to each other, and

$$P'CD' = \pi - \angle RQ'r,$$

and is therefore equal to the given angle.

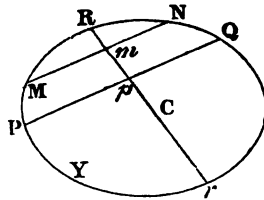
CHAPTER IV.

MISCELLANEOUS PROPOSITIONS.

208. *AN ellipse being traced upon a plane, to find its centre and axes.*

(1) *To find its centre.*

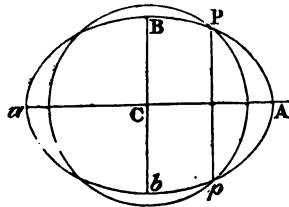
Draw any two parallel chords, MN , PQ , and bisect them



in the points m , p respectively; join m , p and produce mp to meet the ellipse in R , r ; then, because mp passes through the middle points of two parallel chords, it is a diameter (162), and therefore C , the middle point of Rr , is the centre required.

(2) *To find the axes.*

Assume any point P in the ellipse, and from the point C ,

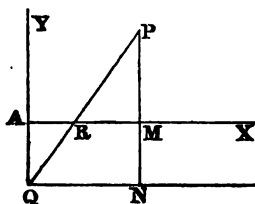


just found, as centre, with distance CP , describe a circle, cutting the ellipse in p , draw Pp and bisect it at right angles by a

straight line ACa meeting the ellipse in A and a ; then ACa is the *major axis*; and the *minor axis* is obtained by drawing BCb at right angles to Aa , meeting the ellipse in the points B, b .

209. *To find the locus of the extremity of a straight line, which moves on two straight lines at right angles to each other, so that the parts intercepted by these lines may always be of the same given length.*

Let AX, AY be the given straight lines, QRP any position



of the straight line, the locus of whose extremity is sought.

Assuming AX, AY as the axes of co-ordinates, let fall the perpendicular PM on AX , and produce it to meet, in N , a parallel to AX drawn through the point Q .

Let $AM = x$, $MP = y$, $QP = a$, $PR = b$,

then $QP^2 = QN^2 + NP^2 \dots \dots \dots (1)$;

but $QN = AM = x$,

and $NP = \frac{QP}{RP} \cdot MP = \frac{a}{b} y$,

therefore, by substitution in (1),

$$a^2 = x^2 + \frac{a^2}{b^2} y^2,$$

$$\text{or } a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the equation to an ellipse.

Therefore the locus of P is an ellipse, of which A is the centre, and $2a$ and $2b$ the axes.

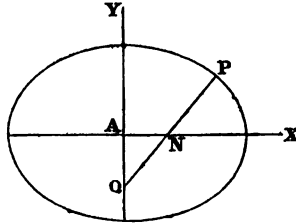
210. COR. 1. By a method precisely similar to the above it may be proved, that if the lines AX and AY are inclined to each other at an angle θ , the equation to the locus of P will be

$$a^2y^2 + b^2x^2 + 2ab \cos \theta . xy - a^2b^2 = 0,$$

which represents an ellipse, referred to any two diameters whatever.*

211. COR. 2. Hence may be derived an easy practical method of describing an ellipse, by means of an instrument called the *Elliptic Compasses*.

Let QP be assumed equal to the semi-major, and NP to



the semi-minor, axis; and let the straight line QNP be turned round, so that the points Q, N may always remain upon the axes of co-ordinates; then the point P will describe an ellipse, as is evident from the foregoing investigation.

212. *In the major-axis Aa of an ellipse, to find a point O, such, that if any chord whatever POP be drawn through it, the angle PAp may be a right angle.*

As in the case of the parabola (p. 86), the co-ordinates of P and p may be determined, by eliminating y between each of the equations

$$y = ax, \quad y = -\frac{1}{a}x,$$

and the equation to the ellipse,

* Hamilton's *Analytical Geometry*, Arts. 89 and 91.

$$y^2 = \frac{b^2}{a^2} (2ax - x^2),$$

we thus have, employing the same notation as in the Article referred to,

$$x' = \frac{2b^2a}{a^2a^2 + b^2}, \quad y' = \frac{2b^2aa}{a^2a^2 + b^2},$$

$$x'' = \frac{2b^2aa^2}{a^2 + b^2a^2}, \quad y'' = -\frac{2b^2aa}{a^2 + b^2a^2};$$

therefore, denoting $2b^2a$ by c , and the denominator in the first and second lines, respectively, by m and n , we have for the equation to Pp ,

$$y - \frac{ca}{m} = \frac{-\frac{ca}{n} - \frac{ca}{m}}{\frac{ca^2}{n} - \frac{c}{m}} \left(x - \frac{c}{m} \right)$$

$$= -a \cdot \frac{m+n}{a^2m-n} \left(x - \frac{c}{m} \right).$$

Let Pp cut the axis, as at O , then $y = 0$, and

$$x - \frac{c}{m} = \frac{c}{m} \cdot \frac{a^2m-n}{m+n};$$

$$\therefore x = c \frac{a^2+1}{m+n},$$

$$\text{but } m+n = a^2a^2 + b^2 + b^2a^2 + a^2$$

$$= (a^2 + b^2)(a^2 + 1);$$

$$\therefore x \text{ or } AO = \frac{c}{a^2+b^2} = \therefore \frac{2b^2a}{a^2+b^2}.$$

213. *Pairs of tangents to an ellipse being always supposed to intersect at right angles, to find the locus of the points of intersection.*

If the straight line

$$y = ax + \beta \dots \dots \dots (1),$$

be drawn to cut the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2 \dots \dots \dots (2),$$

the ordinates of the two points of section will be obtained from the equation

$$(a^2 a^3 + b^3) y^2 - 2b^2 \beta y + b^3 (\beta^2 - a^2 a^3) = 0, \text{ Art. 139.}$$

Let the secant be now supposed to become a tangent, then the two roots of this equation are equal, and the equation being therefore a perfect square, four times the product of the extreme terms = the square of the mean;

$$\therefore 4(a^2 a^3 + b^3) b^3 (\beta^2 - a^2 a^3) = 4b^4 \beta^2,$$

$$\text{or } (a^2 a^3 + b^3) (\beta^2 - a^2 a^3) = b^3 \beta^2;$$

$$\therefore a^2 a^3 \beta^2 - a^4 a^4 - b^2 a^2 a^2 = b^3;$$

$$\therefore a^2 a^3 \beta^2 = a^4 a^4 + b^2 a^2 a^2;$$

$$\therefore a^2 a^3 + b^3 = \beta^2 = (y - ax)^2 \text{ from (1)}$$

$$= y^2 - 2axy + a^2 x^2;$$

$$\therefore (a^2 - x^2) a^3 + 2xy \cdot a + b^2 - y^2 = 0,$$

$$\text{or } a^2 + \frac{2xy}{a^2 - x^2} a + \frac{b^2 - y^2}{a^2 - x^2} = 0.$$

Suppose a' , a'' to be the roots of this equation, then they denote the trigonometrical tangents of the angle which the tangents to the ellipse form with the axis of x ; and by the theory of equations

$$aa' = \frac{b^2 - y^2}{a^2 - x^2};$$

but, by hypothesis, the tangents intersect at right angles,

$$\therefore aa' = -1;$$

$$\text{hence } \frac{b^2 - y^2}{a^2 - x^2} = -1;$$

$$\therefore b^2 - y^2 = -a^2 + x^2;$$

$$\text{or } y^2 + x^2 = a^2 + b^2,$$

which is the equation to a circle.

Hence the locus required is a circle whose radius

$$= \sqrt{a^2 + b^2}.$$

214. COR. In the same manner we may find the locus of the intersection of pairs of tangents, which are always parallel to conjugate diameters; in other words, the locus of the vertices of all *the equal* parallelograms circumscribing the ellipse.*

$$\text{For in this case } aa' = -\frac{b^2}{a^2};$$

$$\therefore \frac{b^2 - y^2}{a^2 - x^2} = -\frac{b^2}{a^2};$$

$$\therefore a^2 b^2 - a^2 y^2 = -b^2 a^2 + b^2 x^2;$$

$$\therefore a^2 y^2 + b^2 x^2 = 2a^2 b^2,$$

which is the equation to an ellipse.

Hence the locus required is an ellipse, whose centre is the same with that of the original, and whose axes may thus be found;

$$\text{Let } x = 0; \therefore a^2 y^2 = 2a^2 b^2;$$

$$\therefore y = b\sqrt{2} = \text{the semi-minor axis};$$

and, in like manner,

$$x = a\sqrt{2} = \text{the semi-major axis}.$$

215. Let CP, CD be a given system of conjugate diameters, join D, P, take any point Q in the ellipse, and draw QR parallel to PD, meeting CP in R, and CD in r; it is required to prove that

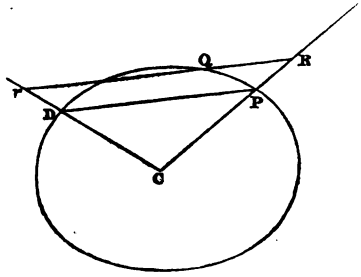
$$QR^2 + Qr^2 = PD^2.$$

* Of all parallelograms circumscribing an ellipse, those only are of equal area, which have their sides parallel to a system of conjugate diameters. See Art. 186.

Let x', y' be the co-ordinates of Q , and let

$$CP = a', \quad CD = b', \quad \text{angle } PCD = \gamma.$$

Then if the ellipse be referred to the given conjugate diameters as axes, the equation to the point D is



$$x = 0, \quad y = b',$$

and that to the point P is

$$y = 0, \quad x = a';$$

therefore the equation to DP is (20)

$$y = \frac{b'}{a'} (a' - x);$$

also, since QR is drawn through a point (x', y') parallel to DP , its equation will be (24)

$$y - y' = \frac{b'}{a'} (x - x');$$

at the point R where this line cuts the axis of x , $y = 0$;

$$\left. \begin{aligned} \therefore x - x' &= -\frac{a'}{b'} y' \\ \text{hence } y - y' &= \frac{b'}{a'} (x - x') = -y' \end{aligned} \right\} \dots\dots(1).$$

Now when the axes of co-ordinates are inclined to each other at an angle γ , the general expression for the distance between any two points is (38)

$$= \sqrt{\{(x - x')^2 + (y - y')^2 + 2(x - x')(y - y') \cos \gamma\}};$$

therefore substituting for $x - x'$, and $y - y'$, their values in (1), we have

$$\begin{aligned} QR^2 &= \frac{a'^2}{b'^2} y'^2 + y'^2 + \frac{2a'}{b'} y'^2 \cos \gamma \\ &= \frac{y'^2}{b'^2} \{a'^2 + b'^2 + 2a'b' \cos \gamma\}. \end{aligned}$$

In like manner it may be proved that

$$Qr^2 = \frac{x'^2}{a'^2} \{a'^2 + b'^2 + 2a'b' \cos \gamma\};$$

$$\begin{aligned} \therefore QR^2 + Qr^2 &= \{a'^2 + b'^2 + 2a'b' \cos \gamma\} \left(\frac{y'^2}{b'^2} + \frac{x'^2}{a'^2} \right) \\ &= a'^2 + b'^2 + 2a'b' \cos \gamma, \text{ Art. 131, (4)} \\ &= \therefore PD^2 \end{aligned}$$

as was to be proved.

216. COR. 1. Hence the sum of these squares is a constant quantity.

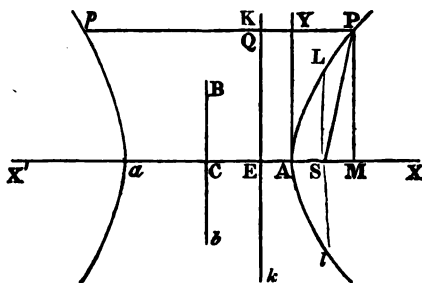
217. COR. 2. Let C be the centre of a circle, CA , CB two radii at right angles to each other; from Q any point in the circumference draw Qr parallel to AB , meeting CA in R , and CB in r ; then

$$QR^2 + Qr^2 = AB^2.$$

CHAPTER I.

218. *To find the equation to the hyperbola.*

Let S be the focus, Kk the directrix, P any point in the hyperbola, through S draw the indefinite line ESX perpen-



Let the given ratio of $PS : PQ$ be as $e : 1$, e being > 1 ; then if SE be divided in A , so that $SE : AE :: e : 1$, A will be a point in the hyperbola.

From A draw AY at right angles to AX , and assume AX and AY as the axes of co-ordinates.

Let $AM = x$, $MP = y$, $AS = m$;

then $SP^2 = PM^2 + MS^2 = y^2 + (x - m)^2 \dots \dots \dots (1)$,

but $SP^2 = e^2 \cdot PQ^2 = e^2 (AE + AM)^2$
 $= e^2 \left(\frac{m}{e} + x \right)^2 \dots \dots \dots (2)$;

therefore equating (1) and (2),

$$y^2 + (x - m)^2 = m^2 + 2mex + e^2 x^2;$$

$$\therefore y^2 = 2m(1 + e)x + (e^2 - 1)x^2$$

$$= (e^2 - 1) \left(\frac{2m}{e - 1} x + x^2 \right),$$

or if $\frac{m}{e - 1}$ be assumed $= a$,

$$y^2 = (e^2 - 1)(2ax + x^2),$$

which is the equation required.

219. COR. 1. In AX' take $Aa = \frac{2m}{e - 1}$, bisect Aa in C , then at this point $x = -a$;

$$\therefore y^2 = - (e^2 - 1) a^2;$$

$$\therefore y = \pm a \sqrt{-1} \cdot \sqrt{e^2 - 1},$$

which is always imaginary, since $e > 1$.

Hence, if BCb be drawn through C at right angles to Aa , and CB , Cb be each taken $= a \sqrt{e^2 - 1}$, the points B and b are *not* points in the hyperbola.

220. COR. 2. Let Bb be denoted by $2b$, then

$$b = \pm a \sqrt{e^2 - 1};$$

$$\therefore \sqrt{e^2 - 1} = \pm \frac{b}{a};$$

therefore, by substitution, the above equation becomes

$$y = \pm \frac{b}{a} \sqrt{2ax + x^2}.$$

DEF. The straight lines Aa , Bb represented by $2a$ and $2b$ are called, respectively, the *transverse*, and the *conjugate*, axis; the points A , a , in which the former meets the hyperbola, are called the *vertices*; and the point C , in which the axes intersect each other, the *centre*.

221. To find the equation to the hyperbola when the co-ordinates originate at the centre.

Let P be any point in the hyperbola, let fall the perpendicular PM on Aa , and assume $CM = x'$.

Then the equation to the hyperbola, when the co-ordinates originate at A , is

$$y^2 = \frac{b^2}{a^2} (2ax + x^2) \dots \dots \dots (1),$$

$$\begin{aligned} \text{but } x &= AM = CM - CA \\ &= x' - a. \end{aligned}$$

Substituting this value for x , we have

$$\begin{aligned} y^2 &= \frac{b^2}{a^2} \{2a(x' - a) + (x' - a)^2\} \\ &= \frac{b^2}{a^2} (x'^2 - a^2) \dots \dots \dots (2), \end{aligned}$$

which is the equation required.

222. **COR. 1.** Suppressing the accent, which was used only to distinguish the new from the old abscissa, we have by multiplying and transposing,

$$a^2 y^2 - b^2 x^2 = -a^2 b^2 \dots \dots \dots (3),$$

and if each term be divided by $a^2 b^2$, we have

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1 \dots \dots \dots (4).$$

Of the three last forms of the equation to the hyperbola, that marked (3) is most frequently used.

223. COR. 2. These equations, when translated into geometrical language, express a property of the hyperbola.

For if P be any point, (Fig. p. 153), we have

$$2ax + x^2 = x(2a + x) = AM \cdot Ma,$$

$$\text{and } x'^2 - a^2 = (x' - a)(x' + a) = AM \cdot Ma;$$

$$\therefore MP^2 = \frac{BC^2}{CA^2} AM \cdot Ma,$$

$$\text{or } AM \cdot Ma : MP^2 :: AC^2 : BC^2,$$

that is, *the rectangle contained by the segments of the transverse axis is to the square of the ordinate, as the square of the semi-transverse axis is to the square of the semi-conjugate.*

224. COR. 3. Let $a = b$, then equations (1) and (2) become

$$y^2 = 2ax + x^2,$$

$$y^2 = x^2 - a^2.$$

The hyperbola represented by these equations is called *equilateral*, or *rectangular*, and is to the common hyperbola, what the circle is to the ellipse.

225. COR. 4. By comparing the equation to the hyperbola

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

with the equation to the ellipse

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

it is manifest, that to pass from the one curve to the other we have only to change $+b^2$ into $-b^2$, or b into $b\sqrt{-1}$,

226. *To determine the figure of the hyperbola from its equation.*

Resuming the equation

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

in which the co-ordinates originate at the centre, we have either

$$y = \pm \frac{b}{a} \sqrt{x^2 + a^2} \dots\dots\dots (1),$$

$$\text{or } x = \pm \frac{a}{b} \sqrt{y^2 + b^2} \dots\dots\dots (2).$$

(1) In equation (1),

$$\text{let } x = 0,$$

$$\text{then } y = \pm b \sqrt{-1},$$

and therefore the hyperbola does not intersect the axis CY .

$$\text{Let } y = 0,$$

$$\text{then } x = \pm a = CA \text{ or } Ca.$$

$$\text{Let } x < \pm a,$$

then the values of y are imaginary; therefore no point of the hyperbola is situated between A and a .

$$\text{Let } x = \pm a,$$

$$\text{then } y = \pm 0;$$

that is, the hyperbola cuts the axis XX' at the points A , a .

$$\text{Let } x > \pm a,$$

then for each value of x , there are two equal values of y , with contrary signs.

As x increases, the values of y increase; and when x becomes indefinitely great, the values of y become so likewise.

The hyperbola, therefore, consists of two opposite branches, extending indefinitely to the *right* of A and to the *left* of a , and symmetrically placed with respect to the axis XX' .

(2) The discussion of equation (2) would lead to the same result.

227. COR. 1. The distance between the centre and focus

$$\begin{aligned}
 &= CS = CA + AS \\
 &= a + a(e - 1) = ae.
 \end{aligned}$$

This quantity is called the *eccentricity* of the hyperbola.

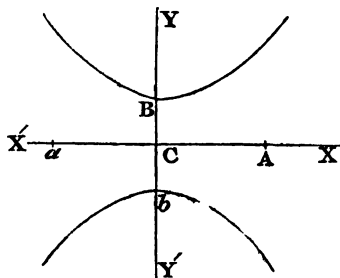
228. COR. 2. To find the value of the ordinate passing through the focus.

When the ordinate passes through the focus,

$$\begin{aligned}
 x &= ae; \\
 \therefore y^2 &= \frac{b^2}{a^2} (a^2 e^2 - a^2) \\
 &= b^2 (e^2 - 1) \\
 &= \frac{b^4}{a^2}; \\
 \therefore y &= \pm \frac{b^2}{a} = SL \text{ or } Sl, \text{ (Fig. p. 153).}
 \end{aligned}$$

The double ordinate Ll , passing through the focus, is called the *principal parameter*, or *latus rectum*.

229. COR. 3. In the equation $a^2 y^2 - b^2 x^2 = -a^2 b^2$,



let x be changed into y , and y into x ; in other words, let the

abscissas be now reckoned along CY , and the ordinates along CX ; we then have, interchanging a and b ,

$$b^2 x^2 - a^2 y^2 = -a^2 b^2,$$

$$\text{or } y^2 = \frac{b^2}{a^2} (x^2 + a^2).$$

$$\text{Let } x = 0; \therefore y = \pm b,$$

$$y = 0; \therefore x = \pm a\sqrt{-1};$$

therefore Bb is now the transverse, and Aa the conjugate, axis.

This hyperbola is called, relatively to the former, the *conjugate hyperbola*.

Hence both hyperbolas are comprised in the equation

$$y^2 = \frac{b^2}{a^2} (x^2 \mp a^2).$$

230. *When the transverse axis is supposed to become infinitely great, the hyperbola passes into a parabola.*

The equation to the hyperbola is

$$\begin{aligned} y^2 &= \frac{b^2}{a^2} (2ax + x^2) \\ &= \frac{2b^2}{a} x + \frac{b^2}{a^2} x^2 \dots\dots\dots (1). \end{aligned}$$

$$\text{Now } b^2 = e^2 a^2 - a^2 \dots\dots\dots (220)$$

$$= (ea + a)(ea - a),$$

$$(ea + a)AS;$$

$$\therefore \frac{b^2}{a} = (e + 1)AS;$$

therefore by substitution in (1),

$$y^2 = 2AS(e + 1)x + \frac{b^2}{a^2} x^2 \dots\dots\dots (2).$$

Let a be now supposed infinite, then $\frac{b^2}{a^2} = 0$, and since $a = \frac{m}{e-1}$, we have $e-1=0$, or $e=1$; therefore by substitution in (2),

$$y^2 = 4 AS \cdot x,$$

which is the equation to the parabola; whence the truth of the proposition. (See Art. 138).

231. *To find the intersection of a straight line with the hyperbola.*

Let the equation to the proposed line be

$$y = ax + \beta \dots \dots \dots (1).$$

Then the co-ordinates of the point or points of intersection with the hyperbola will be obtained, by combining this equation with that to the hyperbola

$$a^2 y^2 - b^2 x^2 = -a^2 b^2 \dots \dots \dots (2).$$

Substituting, then, in (2) the value of x derived from (1), we have

$$a^2 y^2 - b^2 \left(\frac{y - \beta}{a} \right)^2 = -a^2 b^2;$$

$$\therefore (a^2 a^2 - b^2) y^2 + 2b^2 \beta y - b^2 \beta^2 = -a^2 b^2 a^2;$$

$$\therefore y^2 + \frac{2b^2 \beta}{a^2 a^2 - b^2} y + \frac{b^2 (a^2 a^2 - \beta^2)}{a^2 a^2 - b^2} = 0;$$

from this quadratic are obtained two values of y , which, substituted in (1), furnish two corresponding values of x , therefore the co-ordinates required may be determined.

When the two roots of the quadratic are equal, the points of intersection coincide, and the straight line then *touches* the hyperbola; when they are imaginary, the straight line falls entirely without the hyperbola.

Hence it appears, that a straight line cannot cut an hyperbola in more than two points.

DEF. The portion of the straight line contained within the hyperbola is called a *chord*; when the chord passes through the focus, it is called the *focal chord*.

232. To find the equation to a straight line which touches an hyperbola in a given point.

Let x', y' be the co-ordinates of the given point, and x'', y'' those of any other point in the hyperbola, near the first.

Then the equation to the line drawn through these points is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x') \dots \dots \dots (1).$$

But these two points being in the hyperbola, we have

$$a^2 y'^2 - b^2 x'^2 = -a^2 b^2,$$

$$a^2 y''^2 - b^2 x''^2 = -a^2 b^2;$$

therefore by subtraction,

$$a^2 (y''^2 - y'^2) = b^2 (x''^2 - x'^2);$$

$$\therefore \frac{(y'' + y')(y'' - y')}{(x'' + x')(x'' - x')} = \frac{b^2}{a^2};$$

$$\therefore \frac{y'' - y'}{x'' - x'} = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'};$$

therefore equation (1) becomes by substitution

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'} (x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y') ; then $x'' = x', y'' = y'$, and the secant becomes a tangent at the point (x', y') ; hence the equation to the tangent is

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x'),$$

in which x' and y' are the co-ordinates of the point of contact, and x, y the variable co-ordinates of any point whatever in the tangent.

233. COR. This equation may be simplified, for multiplying each side by a^2y' ,

$$\begin{aligned} a^2yy' - a^2y'^2 &= b^2xx' - b^2x'^2; \\ \therefore a^2yy' - b^2xx' &= a^2y'^2 - b^2x'^2 \\ &= -a^2b^2, \end{aligned}$$

which is the equation most commonly used.

When $a = b$, the hyperbola becomes equilateral, and the equation to the tangent is

$$yy' - xx' = -a^2.$$

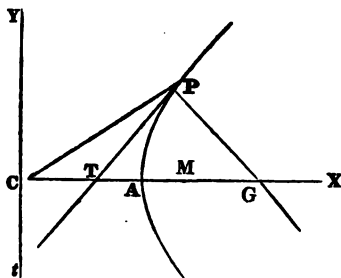
234. To find the intersection of the tangent with the axes of x and y .

The equation to the tangent being

$$a^2yy' - b^2xx' = -a^2b^2.$$

(1) Let it cut the axis of x , as at T ,

$$\text{Then } y = 0; \therefore x = \frac{a^2}{x'},$$



$$\text{or } CT = \frac{CA^2}{CM}.$$

(2) Let it cut the axis of y , as at t ,

$$\text{Then } x = 0; \therefore y = \frac{b^2}{y'},$$

$$\text{or } Ct = \frac{CB^2}{Cm}.$$

Whence it follows, *that each semi-axis is a mean proportional between the abscissa of any point, and the part of it intercepted between its intersection with the tangent and the centre.*

$$235. \text{ COR. Since } CT = \frac{a^2}{x'};$$

$$\therefore MT = CM - CT$$

$$= x' - \frac{a^2}{x'}$$

$$= \frac{x'^2 - a^2}{x'}.$$

DEF. 1. The line MT , intercepted between the foot of the ordinate and the point where the tangent meets the axis, is called the *subtangent*.

DEF. 2. The straight line drawn from the point of contact at right angles to the tangent is called the *normal*.

236. *To find the equation to the normal.*

Let PT touch the hyperbola in P , from which point draw the line PG at right angles to PT .

Then, because PG is drawn through the point (x', y') at right angles to the line

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x')$$

its equation will be (26)

$$y - y' = -\frac{a^2}{b^2} \cdot \frac{y'}{x'} (x - x'),$$

in which x', y' are the co-ordinates of the point of contact, and x, y those of any point whatever in the line PG , considered as indefinite.

237. To find the intersection of the normal with the axes of x and y .

The equation to the normal being

$$y - y' = -\frac{a^2}{b^2} \cdot \frac{y'}{x'} (x - x'),$$

let it intersect

(1) The axis of x , as at G .

$$\text{Then } y = 0, \text{ and } -y' = -\frac{a^2}{b^2} \cdot \frac{y'}{x'} (x - x');$$

$$\therefore x - x' = \frac{b^2}{a^2} x' = MG.$$

(2) The axis of y , as at g^* .

$$\begin{aligned} \text{Then } x = 0; \therefore y - y' &= \frac{a^2}{b^2} \cdot \frac{y'}{x'} x' \\ &= \frac{a^2}{b^2} y'; \\ \therefore y &= \frac{a^2 + b^2}{b^2} y'. \end{aligned}$$

DEF. The line MG intercepted between the foot of the ordinate, and the point where the normal cuts the axis of x , is called the *subnormal*.

The term normal is usually restricted to the finite line PG . See Art. 93.

238. To draw a tangent to an hyperbola from a given point (x'', y'') without it.

Let $x' y'$ be the unknown co-ordinates of the point of contact, then the equation to the tangent being in general

$$a^2 y y' - b^2 x x' = -a^2 b^2,$$

* The point g is not in the figure.

and the point (x'', y'') being, by hypothesis, a point in the tangent, we have

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2 \dots\dots\dots (1);$$

also, the point of contact (x', y') being in the hyperbola,

$$a^2 y'^2 - b^2 x'^2 = -a^2 b^2 \dots\dots\dots (2),$$

hence, by means of these two equations, the co-ordinates x', y' of the point of contact may be determined.

Since the equation which results from elimination between (1) and (2) is of the second degree, it follows, that there are two points of contact; in other words, *that two tangents may be drawn to an hyperbola, from a given point without it.*

But the position of the points of contact may be directly found by constructing, as in Arts. 66, 94, and 147, the loci of equations of (1) and (2), in which x' and y' are variable.

Now the locus of (2) is the given hyperbola, and the locus of (1) is a straight line, whose position is determined by making x' and y' successively = 0.

If, therefore, in the equation

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2,$$

$$x' = 0, \text{ then } y' = -\frac{b^2}{y''},$$

$$y' = 0, \text{ then } x' = \frac{a^2}{x''}.$$

$$\text{Hence if } CN \text{ be taken} = \frac{b^2}{y''} \text{ and } CM = \frac{a^2}{x''},$$

the straight line joining M , N will cut the hyperbola in the points of contact required.

239. COR. 1. The equation to the chord joining the points of contact is

$$a^2 y'' y' - b^2 x'' x' = -a^2 b^2.$$

240. COR. 2. Since CM is independent of y'' , it follows, *that if from the several points of a straight line perpendicular to CX pairs of tangents be drawn to the hyperbola, the chords joining the points of contact, in each case, will all pass through the same given point.*

See Articles 68, 118 and 149.

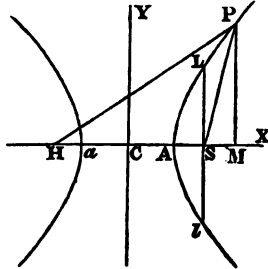
CHAPTER II.

ON THE HYPERBOLA REFERRED TO THE FOCUS.

241. *To find the distance of any point in the hyperbola from either focus.*

Let S, H be the foci, P any point (x, y) in the hyperbola, to find the value of SP , or HP .

(1) Of SP .



In general, the distance between two points (x, y) and (x', y') is (36)

$$= \sqrt{(x - x')^2 + (y - y')^2};$$

but the co-ordinates of S , since it is a point in the axis of x , are $x' = ae, y' = 0$;

$$\begin{aligned} \therefore SP^2 &= (x - ae)^2 + y^2 \\ &= (x - ae)^2 + (e^2 - 1)(x^2 - a^2) \\ &= x^2 - 2aex + a^2e^2 + e^2x^2 - e^2a^2 - x^2 + a^2 \\ &= a^2 - 2aex + e^2x^2; \end{aligned}$$

$$\therefore SP = ex - a.$$

(2) In like manner

$$HP = ex + a.$$

DEF. The distance of any point from the focus is called the *focal distance*.

242. COR. Hence, taking the difference of SP and HP ,

$$HP - SP = 2a.$$

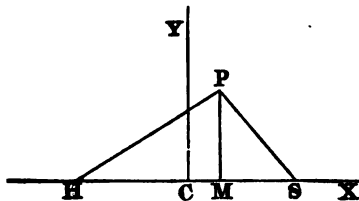
In other words, *the difference of the focal distances is equal to the transverse axis*.

From this property the equation to the hyperbola may be deduced, as in the analogous case of the ellipse, Art. 152.

243. *To find the locus of a point, the difference of whose distances from two fixed points is always equal to a given quantity $2a$.*

Let S, H be the two fixed points, P the point whose locus is required.

Join S, H, S, P , and H, P ; bisect SH in C ; let fall the perpendicular PM on SH , which produce indefinitely towards



X ; from C draw CY at right angles to CX , and assume CX and CY as the axes of co-ordinates.

Let $CM = x$, $MP = y$, and $SC = c$.

Then $SP^2 = SM^2 + MP^2 = y^2 + (c - x)^2$
 $HP^2 = HM^2 + MP^2 = y^2 + (c + x)^2$ (1);

$$\therefore HP^2 - SP^2 = (c + x)^2 - (c - x)^2,$$

$$\text{or } (HP + SP)(HP - SP) = 4cx;$$

$$\text{but } HP - SP = 2a;$$

$$\therefore HP + SP = \frac{4cx}{2a} = \frac{2cx}{a},$$

$$\text{and } HP - SP = 2a;$$

$$\therefore HP = \frac{cx}{a} + a,$$

$$\text{and } SP = \frac{cx}{a} - a,$$

squaring these values, and adding the results,

$$HP^2 + SP^2 = 2 \left(\frac{c^2 x^2}{a^2} + a^2 \right),$$

$$\text{and also } = 2(y^2 + c^2 + x^2)$$

by adding equations (1);

$$\therefore y^2 + c^2 + x^2 = \frac{c^2 x^2}{a^2} + a^2;$$

$$\begin{aligned} \therefore y^2 &= \frac{c^2 x^2}{a^2} + a^2 - c^2 - x^2 \\ &= \frac{x^2}{a^2} (c^2 - a^2) + (a^2 - c^2) \\ &= \frac{c^2 - a^2}{a^2} (x^2 - a^2), \end{aligned}$$

which is the equation to an hyperbola, whose transverse axis $= 2a$, and conjugate axis $= 2\sqrt{c^2 - a^2}$.

If $x = 0$, then $y^2 = -(c^2 - a^2) = -b^2$, if b be the imaginary ordinate drawn from C .

244. **Cor.** Let the fixed points S' , H' be taken in CY , then if P' be a point, such that

$$H'P' \sim S'P' = 2b,$$

the equation to P' will be

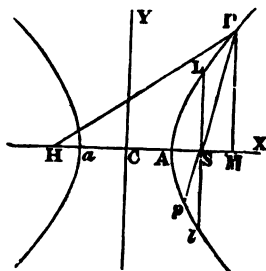
$$y^2 = \frac{b^2}{a^2} (x^2 + a^2),$$

which (229) represents an hyperbola, conjugate to the other.

245. To find the polar equation to the hyperbola, the focus being the pole.

(1) Let S be the pole.

Let $SP = r$, angle $ASP = \omega$.



Then $r = ex - a$, (241),

but $x = CM = CS + SM$

$$= ae + r \cos (\pi - \omega)$$

$$= ae - r \cos \omega ;$$

$$\therefore r = ae^2 - er \cos \omega - a ;$$

$$\therefore r = a \frac{e^2 - 1}{1 + e \cos \omega} ,$$

which is the equation required.

(2) Let H be the pole.

Let $HP = r'$, angle $PHA = \omega'$;

then $r' = ex + a$,

but $x = CM = HM - HC$

$$= r' \cos \omega' - ae ;$$

$$\therefore r' = er' \cos \omega' - ae^2 + a ;$$

$$r' = -a \frac{e^2 - 1}{1 - e \cos \omega},$$

which is the equation required.

See *Observation*, Art. 153.

246. COR. 1. Produce PS to meet the hyperbola in p , then since $\angle Sp = \pi - \omega$,

$$Sp = a \frac{e^2 - 1}{1 - e \cos \omega}.$$

247. COR. 2. Hence

$$\begin{aligned} \frac{1}{SP} + \frac{1}{Sp} &= \frac{1 + e \cos \omega}{a(e^2 - 1)} + \frac{1 - e \cos \omega}{a(e^2 - 1)} \\ &= \frac{2}{a(e^2 - 1)} \\ &= \frac{2}{SL}, \end{aligned}$$

that is, the principal semi-parameter is an harmonic mean between the segments of any chord drawn through the focus.

248. COR. 3. Since $\frac{1}{SP} + \frac{1}{Sp} = \frac{SP + Sp}{SP \cdot Sp}$,

$$\text{and also} = \frac{2}{a(e^2 - 1)};$$

$$\therefore SP \cdot Sp = \frac{1}{2} a (e^2 - 1) (SP + Sp).$$

249. To find the polar equation to the hyperbola, the centre being the pole.

Conceive the points C , P joined (Fig. p. 170), and let $CP = \rho$, and the angle $PCA = \nu$.

Then $\rho^2 = x^2 + y^2$

$$= x^2 + (e^2 - 1)(x^2 - a^2), \text{ (Arts. 220 and 221,)}$$

$$= e^2 x^2 - a^2(e^2 - 1)$$

$$= e^2 \rho^2 \cos^2 \nu - a^2(e^2 - 1);$$

$$\therefore \rho^2 (e^2 \cos^2 \nu - 1) = a^2(e^2 - 1);$$

$$\therefore \rho = a \sqrt{\frac{e^2 - 1}{e^2 \cos^2 \nu - 1}},$$

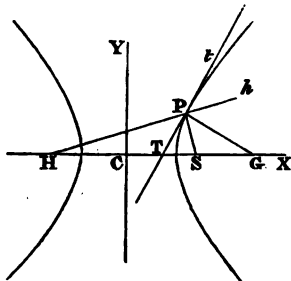
which is the equation required.

250. *The focal distances of any point make equal angles with the tangent at that point.*

Let TPt be a tangent at any point $P(x', y')$; draw the normal PG , and join S , P and H , P .

$$\text{Then } CG = \frac{a^2 + b^2}{a^3} x' \quad (237) = e^2 x' \quad (220);$$

$$\begin{aligned} \therefore \frac{SG}{HG} &= \frac{CG - CS}{CG + CS} = \frac{e^2 x' - ae}{e^2 x' + ae} \\ &= \frac{ex' - a}{ex' + a} = \frac{SP}{HP} \quad (241), \end{aligned}$$



therefore PG bisects the angle SPh . (Euc. vi. Prop. A.)

Now the right angle $GPT = GPt$,

and the angle $GPS = GPh$,

therefore the remaining angle $SPT = hPt = HPT$, that is, SP and HP make equal angles with the tangent at P , as was to be proved.

251. *To find the locus of the points, in which a perpendicular from the focus upon the tangent at any point, intersects the tangent.*

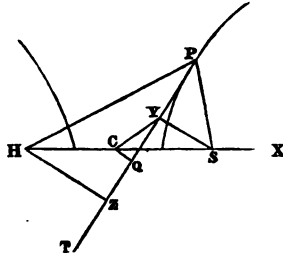
Let PT be a tangent at any point $(x' y')$, and SY a perpendicular let fall from S on PT ; to find the locus of Y .

From C let fall the perpendicular CQ on the tangent, and join S , P , H , P , and C , Y .

$$\text{Then } CY^2 = CQ^2 + QY^2 \dots \dots \dots (1).$$

But since

$$SC = CH,$$



it is plain that

$$YQ = QZ,$$

$$\text{and } CQ = \frac{1}{2}(HZ - SY).$$

Now if ϕ denote the angle which the focal distances make with the tangent,

$$SY = SP \sin \phi,$$

$$\text{and } HZ = HP \sin \phi;$$

$$\therefore CQ = \frac{1}{2}(HP - SP) \sin \phi = a \sin \phi.$$

$$\begin{aligned} \text{Also } QY &= \frac{1}{2} YZ = \frac{1}{2}(PZ - PY) \\ &= \frac{1}{2}(HP - SP) \cos \phi \\ &= a \cos \phi; \end{aligned}$$

therefore by substitution in (1),

$$\begin{aligned} CY^2 &= a^2(\sin^2 \phi + \cos^2 \phi) \\ &= a^2; \end{aligned}$$

$$\therefore CY = \pm a;$$

wherefore the locus of Y is a circle described on the transverse axis.

252. *The rectangle contained by the perpendiculars let fall from the foci upon the tangent at any point is equal to the square of the semi-conjugate axis.*

Let YS be conceived to meet the circumference again in Y' .

$$\text{Let } SY = r, \quad HZ = r', \quad YSX = \theta.$$

Then, if S be the pole, the polar equation to the circle, which is the locus of Y , will be (52),

$$r^2 + 2aer \cos \theta - a^2(e^2 - 1) = 0 \dots\dots\dots(1).$$

Similarly, if H be the pole,

$$r'^2 - 2aer' \cos \theta - a^2(e^2 - 1) = 0 \dots\dots\dots(2).$$

In equation (1), let $\cos \theta$ be changed into $-\cos \theta$, then we have for determining SY' ,

$$r^2 - 2aer \cos \theta - a^2(e^2 - 1) = 0,$$

which is identical with equation (2);

$$\therefore SY' = HZ,$$

but by the theory of equations,

$$SY \cdot SY' = a^2(e^2 - 1);$$

$$\therefore SY \cdot HZ = b^2,$$

as was to be proved.

See *Observation*, Art. 160, and Cor. 161.

CHAPTER III.

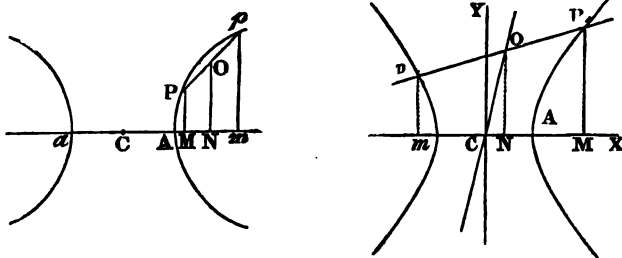
ON THE HYPERBOLA REFERRED TO ANY SYSTEM OF CONJUGATE DIAMETERS.

SECTION I.

ON CONJUGATE DIAMETERS IN GENERAL.

253. *To find the locus of the middle points of any number of parallel chords.*

Let Pp be any chord, O its middle point; from the points O, P, p let fall the perpendiculars ON, PM, pm on the axis AX .



Let $CN = X$, $NO = Y$;

then if the equation to Pp be

$$y = ax + \beta \dots \dots \dots (1),$$

the equation containing the values of y at the points P, p will be (231),

$$y^2 + \frac{2b^2\beta}{a^2\alpha^2 - b^2}y + b^2 \frac{a^2\alpha^2 - \beta^2}{a^2\alpha^2 - b^2} = 0.$$

Now since in any quadratic equation the coefficient of the second term, with its proper sign, is equal to the sum of the roots with their signs changed,

$$\frac{2b^2\beta}{a^2\alpha^2 - b^2} = -(PM + pm).$$

But O being the middle point of Pp ,

$$ON = \frac{PM + pm}{2};$$

$$\therefore Y = \frac{-b^2\beta}{a^2\alpha^2 - b^2} \dots\dots\dots(2).$$

Now X and Y satisfy equation (1), since they are the co-ordinates of a point in Pp ; therefore

$$\begin{aligned} X &= \frac{1}{a}(Y - \beta) \\ &= \therefore \frac{-a^2\alpha\beta}{a^2\alpha^2 - b^2} \dots\dots\dots(3). \end{aligned}$$

To obtain the relation between X and Y , we must eliminate β between (2) and (3);

$$\therefore \frac{a^2\alpha^2 - b^2}{b^2} Y = \frac{a^2\alpha^2 - b^2}{a^2\alpha} X;$$

$$\therefore Y = \frac{b^2}{a^2\alpha} X.$$

Now α remains the same for all chords parallel to Pp (25), therefore the equation just found expresses the relation between the co-ordinates of their middle points, and being of the first degree, the locus required is a *straight line*.

DEF. The straight line, which has been proved to be the locus of the middle points of any number of parallel chords, is

called a *diameter*, and the points in which it intersects the curve are called the *vertices*.

The letters X and Y are introduced to distinguish the two sets of co-ordinates, and the equation to the diameter bisecting any chord,

$$y = ax + \beta,$$

may always be written

$$y = \frac{b^2}{a^2 a} x.$$

From the form of this it is plain that every diameter passes through the centre.

254. *To find the intersection of any diameter with the hyperbola.*

The equation to any diameter being

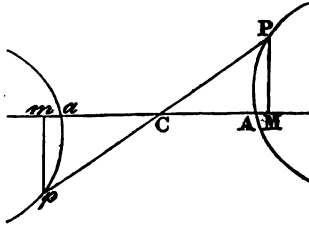
$$y = ax,$$

and that to the hyperbola,

$$a^2 y^2 - b^2 x^2 = -a^2 b^2,$$

the co-ordinates of the points of intersection will be obtained by combining these two equations; we thus have

$$(a^2 a^2 - b^2) x^2 = -a^2 b^2;$$



$$\therefore x = \pm \frac{ab}{\sqrt{b^2 - a^2 a^2}} = CM \text{ or } Cm,$$

$$\text{and } \therefore y = \pm \frac{aba}{\sqrt{b^2 - a^2 a^2}} = PM \text{ or } pm,$$

the co-ordinates required.

255. COR. 1. In the conjugate hyperbola

$$a^2 y^2 - b^2 x^2 = a^2 b^2 ;$$

$$\therefore x = \pm \frac{ab}{\sqrt{a^2 y^2 - b^2}} .$$

256. COR. 2. In order that the diameter may meet the hyperbola,

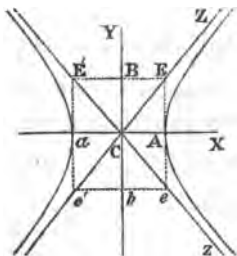
$$b^2 \text{ must be } > a^2 \alpha^2 ,$$

$$\text{or } \pm b \text{ must be } > a \alpha ,$$

$$\text{therefore } \alpha \text{ must be } < \pm \frac{b}{a} .$$

In the same manner α must be $> \pm \frac{b}{a}$, in order that the diameter may meet the conjugate hyperbola.

From the vertex A draw AE and Ae at right angles to AC ,



and each equal to CB ; join C, E, C, e , and produce them indefinitely towards Z and x .

$$\text{Then since } \tan ZCX = \frac{EA}{AC} = \frac{b}{a} ,$$

$$\text{and } \tan xCX = \frac{eA}{AC} = -\frac{b}{a} ,$$

it follows that the diameters CZ, Cx will never meet the curve at any finite distance.

The lines CZ, Cx are, from this circumstance, called *asymptotes*.

257. *A diameter being drawn through a given point, to find the equation to any one of its ordinates.*

If x', y' be the co-ordinates of the given point, the equation to the diameter drawn through it will be

$$y = \frac{y'}{x'} x \dots \dots \dots (1).$$

$$\text{Let } y = ax + \beta \dots \dots \dots (2),$$

be the required equation to any ordinate, then

$$\frac{y'}{x'} = \frac{b^2}{a^2 a} (253);$$

$$\therefore a = \frac{b^2}{a^2} \cdot \frac{x'}{y'},$$

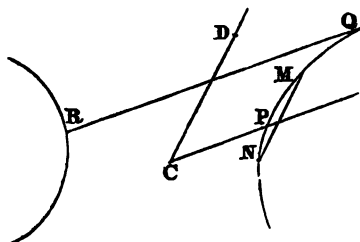
therefore the equation to any ordinate to a diameter passing through (x', y') is

$$y = \frac{b^2}{a^2} \cdot \frac{x'}{y'} x + \beta.$$

258. COR. Comparing this equation with the equation to the tangent (240), it appears, *that the tangent applied at the vertex of any diameter is parallel to the ordinates of that diameter.*

259. *Any two diameters being given, if the ordinates of the one be parallel to the other, the ordinates of the latter will be parallel to the former.*

Let CP, CD be two diameters, and MN, QR chords bi-



sected by each respectively; then if MN be supposed parallel to CD , we are to prove that QR will be parallel to CP .

The equations to CP , CD being

$$y = ax \dots \dots \dots (1),$$

$$y = a'x \dots \dots \dots (2),$$

the equations to MN , QR , respectively, will, by Art. (257), be

$$y = \frac{b^2}{a^2a}x + \beta \dots \dots \dots (1'),$$

$$y = \frac{b^2}{a^2a'}x + \beta' \dots \dots \dots (2').$$

But if MN be parallel to CD , then

$$a' = \frac{b^2}{a^2a} \quad (25);$$

$$\therefore a = \frac{b^2}{a'^2a'},$$

therefore by substitution in (2') the equation to QR becomes

$$y = ax + \beta',$$

that is, (25) QR is parallel to CP , as was to be proved. *Whence each of the diameters CP , CD , is parallel to the ordinates of the other.*

Diameters, thus related to each other, are called *conjugate diameters*.

260. COR. 1. Therefore when the two diameters

$$y = ax,$$

$$y = a'x,$$

are conjugate to each other,

$$aa' = \frac{b^2}{a^2}.$$

261. COR. 2. Hence if

$$y = ax.$$

be any diameter,

$$y = \frac{b^2}{a^2 \alpha} x$$

will be the diameter conjugate to it.

262. COR. 3. Since α may have any value between 0 and π , the number of pairs of conjugate diameters is infinite.

If $\alpha = 0$, or the first diameter be the transverse axis Aa , then

$$\alpha' = \frac{b^2}{a^2 \cdot 0} = \infty,$$

therefore the diameter conjugate to Aa being at right angles to it, is the conjugate axis Bb ; whence the axes are conjugate diameters; and it may be shewn, precisely as in Art. 170, that they are the only conjugate diameters which are at right angles to each other.

263. COR. 4. If (x', y') be any point in the hyperbola, the diameter passing through it is

$$y = \frac{y'}{x'} x;$$

$$\therefore y = \frac{b^2 x'}{a^2 y'} x.$$

in the corresponding conjugate diameter.

But the equation to a tangent applied at the point (x', y') is (232)

$$y - y' = \frac{b^2 x'}{a^2 y'} (x - x'),$$

whence it follows (25), that *the tangent at the vertex of any diameter is parallel to the corresponding conjugate diameter.*

264. *Of any two conjugate diameters, only one can meet the hyperbola.*

For let $y = \alpha x$,

$$y = \alpha' x,$$

be any two conjugate diameters.

It was shewn (254) that no diameter can meet the hyperbola unless

$$a < \frac{b}{a}.$$

Suppose, then, in the given system, that the first diameter meets the hyperbola, then

$$a < \frac{b}{a},$$

$$\text{but } a\alpha' = \frac{b^2}{a^2} \text{ (260);}$$

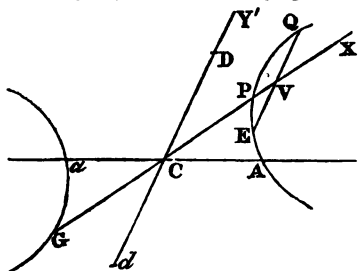
$$\therefore \alpha' > \frac{b}{a},$$

and consequently the second diameter *cannot meet the hyperbola*.

265. Hence it appears that the conjugate diameter will always meet the conjugate hyperbola.

266. *To find the equation to the hyperbola, when it is referred to any two conjugate diameters as axes.*

Let CP , CD , any system of conjugate diameters, be as-



sumed as axes of co-ordinates, and let x, y and x', y' be the co-ordinates of any point in the hyperbola, when it is referred to the old and the new axes respectively.

The object now is, to express x, y in terms of x', y' , and then to substitute the resulting values in the equation

$$a^2y^2 - b^2x^2 = -a^2b^2 \dots\dots\dots (1),$$

which will give the equation required.

It may be shewn, precisely as in the ellipse (171), that if $\theta = X'CA$ and $\phi = Y'CA$,

$$x = x' \cos \theta + y' \cos \phi,$$

$$y = x' \sin \theta + y' \sin \phi;$$

therefore by substitution in (1),

$$a^2 \{x' \sin \theta + y' \sin \phi\}^2 = b^2 \{x' \cos \theta + y' \cos \phi\}^2 = -a^2 b^2,$$

$$\text{or } (a^2 \sin^2 \phi - b^2 \cos^2 \phi) y'^2 + (a^2 \sin^2 \theta - b^2 \cos^2 \theta) x'^2$$

$$+ 2x'y' (a^2 \sin \theta \sin \phi - b^2 \cos \theta \cos \phi) = -a^2 b^2.$$

$$\text{But } \tan \theta \tan \phi = \frac{b^2}{a^2} \text{ (256) ;}$$

$$\therefore a^2 \sin \theta \sin \phi - b^2 \cos \theta \cos \phi = 0;$$

therefore the term involving $x'y'$ vanishes, and the equation required is of the form

$$(a^2 \sin^2 \phi - b^2 \cos^2 \phi) y'^2 - (b^2 \cos^2 \theta - a^2 \sin^2 \theta) x'^2 = -a^2 b^2 \dots (2).$$

Now when the axis CX' meets the hyperbola,

$$y' = 0, \text{ and } x' = CP = a';$$

$$\therefore b^2 \cos^2 \theta - a^2 \sin^2 \phi = \frac{a^2 b^2}{a'^2}.$$

Let $x' = 0$, then since the axis CY' does not meet the hyperbola (264), we may assume y' or $CD = b'\sqrt{-1}$;

$$\therefore a^2 \sin^2 \phi - b^2 \cos^2 \phi = \frac{-a^2 b^2}{-b'^2} = \frac{a^2 b^2}{b'^2};$$

therefore by substitution in (2),

$$\frac{a^2 b^2}{b'^2} y'^2 - \frac{a^2 b^2}{a'^2} x'^2 = -a^2 b^2,$$

or, dividing each term by $a^2 b^2$,

$$\frac{y'^2}{b'^2} - \frac{x'^2}{a'^2} = -1 \dots \dots \dots (1),$$

$$\text{or, } a'^2 y'^2 - b'^2 x'^2 = -a'^2 b'^2 \dots \dots \dots (2),$$

either of which is the equation required.

267. COR. 1. Hence, suppressing the accents of the co-ordinates,

$$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2}.$$

268. COR. 2. To find the form of the equation, when the co-ordinates originate at P , the vertex of the diameter CP .

Let $PV = a'$, then $x = CP + PV = a' + x'$.

Substituting this value of x in Cor. 1, we have

$$y = \pm \frac{b'}{a'} \sqrt{(x' + a')^2 - a'^2},$$

or, suppressing the accent of x ,

$$y = \pm \frac{b'}{a'} \sqrt{x^2 + 2a'x} \dots \dots \dots (3),$$

which is the equation required.

269. COR. 3. The equations (1), (2), and (3), are of the same form with the equations in terms of the axes (Arts. 221 and 222), and express a property of the hyperbola;

$$\text{For } x^2 - a'^2 = (x + a')(x - a') = PV \cdot VG,$$

$$\text{and } 2a'x + x^2 = (2a' + x)x = PV \cdot VG;$$

$$\therefore VQ^2 = \frac{PC^2}{CD^2} PV \cdot VG,$$

$$\text{or } PV \cdot VG : QV^2 :: PC^2 : CD^2;$$

that is, *the rectangle contained by the segments of any diameter is to the square of the ordinate, as the square of the semi-diameter is to the square of its semi-conjugate.*

270. It appears from the preceding Article, that the equation to the hyperbola retains the same form, whether the axes of co-ordinates be rectangular or oblique. Whence it follows, when the axes are oblique,

(1) That if the equation to the transverse axis Aa be

$$y = ax,$$

the equation to the conjugate axis Bb will be

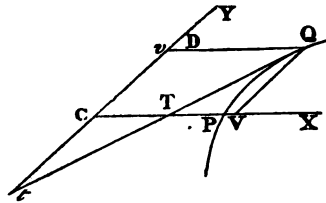
$$y = \frac{b'^2}{a'^2 a} x.$$

(2) That the equation to the tangent at any point (x', y') is

$$a'^2 y y' - b'^2 x x' = -a'^2 b'^2.$$

271. *To find the intersection of the tangent with any two conjugate diameters.*

Let CP, CD , be any two conjugate diameters, and let a tangent applied at any point $Q(x', y')$ meet CP in T ,



and CD in t , and draw the ordinates QV, Qv , to CP, CD respectively. Then if CP, CD be assumed as axes, the equation to the tangent will be

$$a'^2 y y' - b'^2 x x' = -a'^2 b'^2,$$

Let the tangent meet CX as at T , then $y = 0$;

$$\text{and } x = \frac{a'^2}{x'}, \text{ or } CT = \frac{CP^2}{CV}.$$

Let the tangent meet CY as at t , then $x = 0$;

$$\text{and } y = \frac{-b'^2}{y'}, \text{ or } Ct = \frac{CD^2}{Cv}.$$

Whence the points of intersection are known.

See Art. 234, which is only a particular case of this Article.

272. If from the several points of a straight line given in position pairs of tangents be drawn to an hyperbola, the straight lines which join the corresponding points of contact will all pass through the same point.

Let C be the centre of the hyperbola, MN the given line, draw any chord mn parallel to MN , and bisect it by the diameter CX ; from C draw CY parallel to mn , or MN , then CX , CY are conjugate diameters (259), and if the hyperbola be referred to these as axes, its equation will be

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \dots\dots\dots (1).$$

From any point (x'', y'') in MN , let a pair of tangents be drawn to the hyperbola, then it may be shewn, as in Art. 238, that the equation to the line joining the points of contact is

$$a'^2 y'' y' - b'^2 x'' x' = -a'^2 b'^2 \dots\dots\dots (2),$$

in which x' , y' are the variable co-ordinates of the point of contact.

Let the straight line (2) cut the axis of x , then $y' = 0$, and

$$\therefore x' = \frac{a'^2}{x''};$$

hence the point of intersection will be the same for all points whose abscissas are equal to x'' , that is, for all points in the straight line MN , as was to be proved.

273. Cor. The point of intersection is situated on the diameter, conjugate to that which is parallel to the given line.

274. *If from the point of intersection of two tangents a diameter be drawn, it will bisect the line joining the points of contact.*

For the equation to an ordinate to the diameter passing through (x'', y'') is (253)

$$y = \frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x + \beta \dots \dots \dots (1),$$

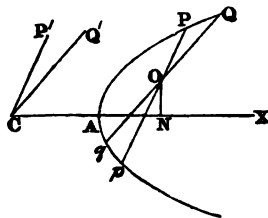
and the equation to the straight line joining the points of contact is

$$y' = \frac{b'^2}{a'^2} \cdot \frac{x''}{y''} x' + \frac{b'^2}{y''} \dots \dots \dots (2);$$

therefore the latter being parallel to the former (25), is also an ordinate, and is therefore bisected.

275. *If through any point within or without an hyperbola two straight lines, given in position, be drawn to meet the curve, the rectangle contained by the segments of the one will bear a constant ratio to the rectangle contained by the segments of the other.*

From O (α, β) let fall the perpendicular ON upon AX .



Let $OP = r$, $OQ = r'$,

and denote by θ, θ' the angles which r, r' make with AX .

Then referring the figure to rectangular co-ordinates,

$$a^2 y^2 - b^2 x^2 = -a^2 b^2 \dots \dots \dots (1).$$

$$\text{But } y = \beta + r \sin \theta,$$

$$x = \alpha + r \cos \theta;$$

therefore by substitution in (1),

$$a^2 \{\beta + r \sin \theta\}^2 - b^2 \{\alpha + r \cos \theta\}^2 = -a^2 b^2,$$

or, developing, arranging the result, and dividing each term by the coefficient of r^2 ,

$$r^2 + \frac{2\{a^2 \beta \sin \theta - b^2 \alpha \cos \theta\}}{a^2 \sin^2 \theta - b^2 \cos^2 \theta} r + \frac{a^2 \beta^2 - b^2 \alpha^2 + a^2 b^2}{a^2 \sin^2 \theta - b^2 \cos^2 \theta} = 0,$$

in which the values of r are OP, Op ; therefore by the theory of equations,

$$OP \cdot Op = \frac{a^2 \beta^2 - b^2 \alpha^2 + a^2 b^2}{a^2 \sin^2 \theta - b^2 \cos^2 \theta}.$$

In like manner

$$OQ \cdot Oq = \frac{a^2 \beta^2 - b^2 \alpha^2 + a^2 b^2}{a^2 \sin^2 \theta' - b^2 \cos^2 \theta'};$$

$$\therefore OP \cdot Op : OQ \cdot Oq :: a^2 \sin^2 \theta' - b^2 \cos^2 \theta' : a^2 \sin^2 \theta - b^2 \cos^2 \theta,$$

and as θ, θ' are known, these rectangles are to each other in a constant ratio, as was to be proved.

276. COR. 1. It may be proved, precisely as in the case of the ellipse, Art. 181, that if CP', CQ' be semi-diameters of the conjugate hyperbola, parallel to Pp, Qq respectively,

$$OP \cdot Op : OQ \cdot Oq :: CP'^2 : CQ'^2.$$

COR. 2. Hence also, if the point O be without the hyperbola, and the points P, p and Q, q coincide, then OP, OQ will become tangents, and

$$OP^2 : OQ^2 :: CP^2 : CQ^2,$$

$$\text{or } OP : OQ :: CP' : CQ'.$$

COR. 3. If Pp be supposed to pass through the centre, and to bisect Qq in O , then the semi-diameter which is parallel to OQ will be conjugate to CP , and we shall have

$$PO \cdot Op : OQ^2 :: CP^2 : CD^2,$$

as in Art. 269.

SECTION II.

ON THE PROPERTIES OF CONJUGATE DIAMETERS.

277. *A DIAMETER being drawn through a given point (x', y') , to find the co-ordinates of the extremity of the diameter conjugate to it.*

Let CP, CD be any two conjugate diameters, of which the former is drawn through the given point $P(x', y')$; then the latter CD will meet only the conjugate hyperbola (267).

$$\text{Let } y = \frac{y'}{x'} x \dots \dots \dots (1),$$

be the equation to CP , then

$$y = \frac{b^2 x'}{a^2 y'} x \dots \dots \dots (2),$$

will be the equation to CD ; therefore the co-ordinates of the point D , will be found by combining (2) with the equation to the conjugate hyperbola

$$a^2 y^2 - b^2 x^2 = a^2 b^2 \dots \dots \dots (3).$$

Hence, substituting in (3) the value of y in (2), and dividing the result by b^2 , we have

$$\left(\frac{b^2 x'^2}{a^2 y'^2} - 1 \right) x^2 = a^2;$$

$$\therefore (b^2 x'^2 - a^2 y'^2) x^2 = a^4 y'^2.$$

$$\text{or } a^2 b^2 x^2 = a^4 y'^2;$$

$$\therefore x = \pm \frac{a}{b} y';$$

$$\begin{aligned} \therefore y &= \frac{b^2 x'}{a^2 y'} x \\ &= \pm \frac{b}{a} x'. \end{aligned}$$

278. *The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the semi-axes.*

Let CP , CD be any two semi-conjugate diameters, then denoting them by a' and b' respectively,

By the last Article,

$$x^2 + y^2 \text{ or } b'^2 = \frac{a^2}{b^2} y'^2 + \frac{b^2}{a^2} x'^2 \dots\dots\dots (1),$$

$$\text{but } y'^2 = \frac{b^2}{a^2} (x'^2 - a^2) \text{ and } x'^2 = \frac{a^2}{b^2} (b^2 + y'^2);$$

$$\therefore \text{ by substitution in (1), } b'^2 = x'^2 - a^2 + y'^2 + b^2,$$

$$\text{also } a'^2 = x'^2 + y'^2;$$

$$\therefore a'^2 - b'^2 = a^2 - b^2,$$

as was to be proved.

279. *If at the extremities of any two conjugate diameters tangents be applied so as to form a parallelogram, the area of all such parallelograms is constant.*

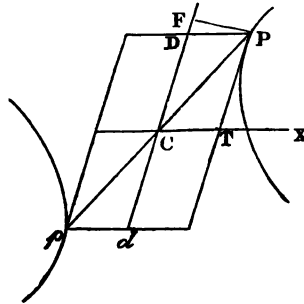
From P let fall the perpendicular PF upon CD produced, and let

$$DCX = \theta, \quad PCX = \theta', \quad DCP = \gamma.$$

Then the area of the whole parallelogram

$$\begin{aligned}
 &= 4CD \cdot PF \\
 &= 4CD \cdot CP \sin \gamma \\
 &= 4a'b' \sin \gamma \dots \dots \dots (1).
 \end{aligned}$$

$$\begin{aligned}
 \text{Now } \sin \gamma &= \sin (\theta - \theta') \\
 &= \sin \theta \cos \theta' - \cos \theta \sin \theta'
 \end{aligned}$$



$$= \frac{y}{b'} \cdot \frac{x'}{a'} - \frac{x}{b'} \cdot \frac{y'}{a'},$$

$$\begin{aligned}
 \therefore a'b' \sin \gamma &= yx' - xy' \\
 &= \frac{b}{a} x'^2 - \frac{a}{b} y'^2 \quad (277) \\
 &= \frac{b}{a} x'^2 - \frac{b}{a} (x'^2 - a^2) \\
 &= \frac{b}{a} \{ x'^2 - x'^2 + a^2 \} \\
 &= ab;
 \end{aligned}$$

therefore the area of the parallelogram is constant, as was to be proved*.

280. COR. 1. It thence appears that

$$ab = a'b' \sin \gamma.$$

* The proof which has been given of the analogous proposition in the ellipse Art. 186, may be applied to the hyperbola.

281. COR. 2. Hence the value of PF may be found ;

$$\text{For } PF = \frac{ab}{CD} = \frac{ab}{\sqrt{a'^2 - (a^2 - b^2)}}.$$

282. COR. 3. Since $a^2 - b^2 = a'^2 - b'^2$, the conjugate diameters cannot be equal to each other in the hyperbola.

283. *The rectangle contained by the focal distances of any point is equal to the square of the semi-diameter, conjugate to that which passes through the proposed point.*

Let P be any point, CD the semi-diameter conjugate to CP , join P, S and P, H ; to prove that

$$SP \cdot HP = CD^2.$$

$$\text{Since } CP^2 - CD^2 = a^2 - b^2;$$

$$\begin{aligned} \therefore CD^2 &= CP^2 - a^2 + b^2 \\ &= x^2 + y^2 - a^2 + b^2 \\ &= x^2 + (e^2 - 1)(x^2 - a^2) - a^2 + b^2 \\ &= e^2 x^2 - e^2 a^2 + b^2 \\ &= e^2 x^2 - a^2 \\ &= (ex - a)(ex + a) \\ &= \therefore SP \cdot HP \text{ (241).} \end{aligned}$$

284. COR. Referring to Art. 251,

$$\text{Since } SP = \frac{SY}{\sin \phi}, \text{ and } HP = \frac{HZ}{\sin \phi};$$

$$\therefore SP \cdot HP = \frac{SY \cdot HZ}{\sin^2 \phi},$$

$$\text{or } b'^2 = \frac{b^2}{\sin^2 \phi};$$

$$\therefore \sin \phi = \frac{b}{b'} \dots\dots\dots(1).$$

Also, since $ab = a'b' \sin \gamma$;

$$\therefore \frac{b}{b'} = \frac{a'}{a} \sin \gamma = \sin \phi,$$

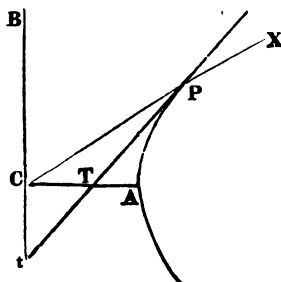
$$\frac{\sin \phi}{\sin \gamma} = \frac{a'}{a} \dots\dots\dots(2).$$

285. Let CP, CD be any two semi-conjugate diameters, and let a tangent at P meet the axes of the hyperbola in T, t ; to prove that $PT \cdot Pt = CD^2$.

If CP, CD be assumed as the axes of the co-ordinates, then the equations to the semi-axes of the hyperbola are respectively (270)

$$y = ax \dots\dots\dots(1),$$

$$y = \frac{b'^2}{a'^2 a} x \dots\dots\dots(2).$$



Let $x = a'$, or CP , then y or $PT = aa'$ from (1),

$$\text{and } y \text{ or } Pt = \frac{b'^2}{a'^2 a} \dots\dots\dots(2);$$

$$\therefore PT \cdot Pt = b'^2 = CD^2.$$

SECTION III.

ON SUPPLEMENTAL CHORDS.

DEF. IF from the vertices of any diameter two straight lines be drawn to any point in the hyperbola, they are called *supplemental chords*.

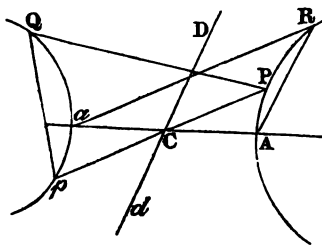
The chords drawn from the vertices of the transverse axis are called the principal supplemental chords.

286. *Any two supplemental chords being drawn, and the equation to either of them being given, to find the equation to the other.*

The hyperbola being referred to any two conjugate diameters, its equation will be

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \dots\dots\dots (1).$$

Through any point $P (x', y')$ draw the diameter Pp , and let



QP, Qp be any two supplemental chords, then if the equation to QP be

$$y - y' = a (x - x') \dots\dots\dots (2),$$

it is required to find the equation to Qp .

The co-ordinates of P being x', y' , those of p will be $-x', -y'$, therefore the equation to pQ will be of the form

$$y + y' = a'(x + x') \dots \dots \dots (3),$$

in which a' is to be found.

Since the straight lines whose equations are (2) and (3) intersect in Q , the co-ordinates of that point will be the same in both; therefore, multiplying them together,

$$y^2 - y'^2 = aa'(x^2 - x'^2);$$

$$\therefore aa' = \frac{y^2 - y'^2}{x^2 - x'^2} \dots \dots \dots (4),$$

but because x', y' are the co-ordinates of P , a point in the hyperbola, they will satisfy the equation (1);

$$\therefore a'^2 y'^2 - b'^2 x'^2 = -a'^2 b'^2.$$

Subtracting this from (1) we have

$$a'^2 (y^2 - y'^2) - b'^2 (x^2 - x'^2) = 0;$$

$$\therefore \frac{y^2 - y'^2}{x^2 - x'^2} = \frac{b'^2}{a'^2};$$

therefore by substitution in (4),

$$aa' = \frac{b'^2}{a'^2}, \text{ and } a' = \frac{b'^2}{a'^2 a},$$

hence the equation to pQ becomes, by substitution in (3),

$$y + y' = \frac{b'^2}{a'^2 a} (x + x').$$

287. COR. 1. Let Pp coincide with the transverse axis Aa , then the equation to aQ drawn through the point a ($-a, 0$) will be

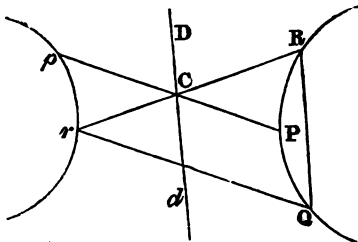
$$y = a(x + a),$$

Let Pp be the given diameter, (Fig. p. 195), and

First, Let the transverse axis be given.

From a draw aR parallel to pP , and join R, A ; then if Dd be drawn through C parallel to RA , it will be conjugate to Pp .

Secondly, if the transverse axis be not given.



Draw any diameter whatever Rr , through r draw rQ parallel to Pp , join Q, R ; then if Dd be drawn through C parallel to RQ , it will be conjugate to Pp .

These conclusions are evident.

291. COR. 2. Hence also is derived a very simple method of applying a tangent at a given point of the hyperbola.

Let P be the given point,* and

First, Let the transverse axis be given.

Draw PC , and the chord aQ parallel to it, join Q, A ; then if PT be drawn parallel to QA , it will touch the hyperbola at P .

Secondly, If the transverse axis be not given.

Draw any diameter RCr , meeting the hyperbola in R, r , join P, C , draw rQ parallel to PC , join Q, R ; then if PT be drawn parallel to QR , it will be a tangent at P .

* The figures in this Cor. may readily be supplied by the Student.

292. *To find the angle contained by the principal supplemental chords.*

Let the point $Q(x', y')$ be the intersection of the chords AQ, aQ (Fig. p. 197), and suppose the hyperbola referred to its axes; then if the equations to Qa, QA be

$$y = a(x + a),$$

$$y = a'(x - a),$$

$$\tan AQA \text{ will be } = \frac{a' - a}{1 + aa'}$$

$$\text{or since } aa' = \frac{b^2}{a^2},$$

$$\tan AQA = \frac{a' - a}{1 + \frac{b^2}{a^2}} \dots\dots\dots(1).$$

$$\text{Now } a' = \tan QAX = \frac{y'}{x' - a},$$

$$\text{and } a = \tan QaX = \frac{y'}{x' + a};$$

$$\therefore a' - a = y' \cdot \left(\frac{1}{x' - a} - \frac{1}{x' + a} \right)$$

$$= y' \cdot \frac{2a}{x'^2 - a^2} = \frac{2ab^2}{a^2 y'},$$

therefore, by substitution in (1)

$$\tan AQA = \frac{2ab^2}{y'(a^2 + b^2)}.$$

293. COR. 1. Since the sign of this quantity is positive, the angle is always acute; therefore, also, the angle contained by any two conjugate diameters is acute.

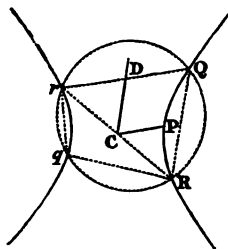
294. COR. 2. When $y' = 0$, $\tan AQA = \infty$, therefore the angle is a right angle; when $y' = \infty$, $\tan AQA = 0$, therefore the angle is $= 0$; hence the acute angle contained by the

supplemental chords in the hyperbola may pass through all degrees of magnitude between 0 and $\frac{\pi}{2}$.

295. *To draw two conjugate diameters making a given angle with each other.*

The analytical solution of the problem is similar to that for the ellipse, except that the reducing equation will be a quadratic, of the fourth degree. We shall therefore proceed to give the geometrical construction.

Draw any diameter Rr , meeting the hyperbola in R, r , and upon it describe a segment of a circle containing an angle equal to the given angle, and cutting the hyperbola in Q , draw QR, Qr , and, parallel, to these, the semi-diameters CP, CD ; these will be the semi-diameters required.



For being parallel to the supplemental chords QR, Qr , they are conjugate to each other (289), and the angle $PCD = RQr$, and is therefore equal to the given angle.

The problem admits also, as in the ellipse, of a second solution. See Art. 207.

In the case of the ellipse, the given angle formed by two conjugate diameters must be confined within certain limits, but in the hyperbola no restriction is necessary (294).

From the principles already laid down, the reader will have no difficulty in deducing such of the miscellaneous propositions on the ellipse, Chap. iv. p. 145, as are applicable to the hyperbola.

CHAPTER IV.

ON THE ASYMPTOTES OF THE HYPERBOLA.

It was shown in Art. 256, that certain diameters of the hyperbola meet the curve only at an infinite distance, and are for that reason termed *Asymptotes*. Since the asymptotes, therefore, pass through the centre, and are inclined to the transverse axis at an angle whose $\tan = \pm \frac{b}{a}$, their equation will be

$$y = \pm \frac{b}{a} x.$$

296. *Let it now be required to find the position of the asymptotes, when the hyperbola is referred to any two conjugate diameters.*

For this purpose it is only necessary to find the intersection of any diameter,

$$y = ax \dots \dots \dots (1),$$

with the hyperbola

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2 \dots \dots \dots (2).$$

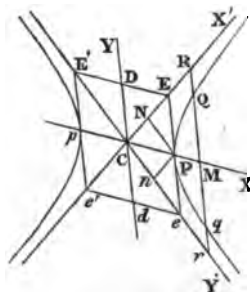
Eliminating y between (1) and (2),

$$(a'^2 a^2 - b'^2) x^2 = -a'^2 b'^2;$$

$$\therefore x = \pm \frac{a' b'}{\sqrt{b'^2 - a'^2 a^2}};$$

$$\text{and } \therefore y = \pm \frac{a' b' a}{\sqrt{b'^2 - a'^2 a^2}}.$$

Now so long as $b'^2 > a'^2 a^2$, or $a < \pm \frac{b'}{a'}$, the diameter meets the curve; but when $a = \pm \frac{b'}{a'}$, the diameter becomes an asymptote.



Hence, if through P the line Ee be drawn, equal and parallel to Dd , and C, E, C, e be joined; the lines CEX' , CeY' , will be asymptotes.

The equation to CX' is $y = \frac{b'}{a'}x$,

and that to CY' is $y = -\frac{b'}{a'}x$.

297. COR. 1. Since Ee touches the hyperbola at P (263), it follows that the part of the tangent intercepted by the asymptote is bisected at the point of contact.

298. COR. 2. If Pn, PN be drawn parallel to CX', CY' , then since $eP = PE$, en will $= nC$, and $CN = NE$.

299. The equation to the asymptotes may be deduced from that to the curve; for we have

$$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2},$$

in which y is the ordinate to the hyperbola, and x the corresponding abscissa. Now in tracing the figure of the hyperbola

from its equation, it was shewn that for each value of x , however great, there are two equal values of y , with contrary signs. If x , therefore, be assumed infinitely great, the ordinate to the curve ought to coincide with that to the asymptote.

Hence in the above equation, expanding the value of y , we have

$$\begin{aligned} y &= \pm \frac{b'}{a'} x \left(1 - \frac{a'^2}{x^2} \right)^{\frac{1}{2}} \\ &= \pm \frac{b'}{a'} x \left(1 - \frac{1}{2} \frac{a'^2}{x^2} - \dots \right) \\ &= \pm \frac{b'}{a'} x \mp \frac{a' b'}{2} \cdot \frac{1}{x} \dots\dots \end{aligned}$$

Let $y = \infty$, therefore all the terms containing x in the denominator vanish, and we have

$$y = \pm \frac{b'}{a'} x,$$

which is the equation required.

300. *The asymptote may be considered as a tangent to the hyperbola, at a point infinitely distant.*

For the equation to a tangent at any point (x', y') is

$$a'^2 y y' - b'^2 x x' = -a'^2 b'^2,$$

$$\text{or } y = \frac{b'^2 x'}{a'^2 y'} x - \frac{b'^2}{y'} \dots\dots\dots(1).$$

$$\text{Now } y' = \pm \frac{b'}{a'} \sqrt{x'^2 - a'^2}.$$

Suppose x' to be infinitely great, then a'^2 vanishes when compared with x'^2 ;

$$\therefore y' = \pm \frac{b'}{a'} x',$$

therefore, by substitution in (1), the equation to the tangent, when the point (x', y') is infinitely distant, becomes

$$y = \pm \frac{b'}{a'} x \pm \frac{a'b'}{x'},$$

$$\text{or since } \frac{a'b'}{x'} = 0,$$

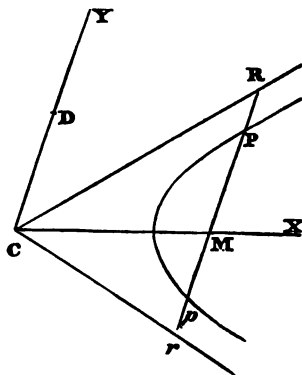
$$y = \pm \frac{b'}{a'} x,$$

which is the equation to the asymptotes: whence the truth of the proposition.

301. *If any chord of the hyperbola be produced to meet the asymptotes, the parts of it intercepted between the curve and the asymptotes will be equal.*

Let the chord Pp be produced to meet the asymptotes in R, r ; to prove that

$$PR = pr.$$



Bisect Pp by the diameter CX , and draw CY conjugate to it; then the equation to the hyperbola being

$$y = \pm \frac{b'}{a'} \sqrt{x^2 - a'^2} \dots\dots\dots$$

that to the asymptotes will be (296)

$$y = \pm \frac{b'}{a} x \dots \dots \dots (2).$$

Now to the same abscissa CM , we have from (1)

$$MP = Mp,$$

and from (2)

$$MR = Mr,$$

therefore by subtraction

$$PR = pr,$$

as was to be proved.

$$\begin{aligned} 302. \quad \text{COR.} \quad \text{Hence } PR \cdot Pr &= (MR - MP)(MR + MP) \\ &= MR^2 - MP^2, \end{aligned}$$

$$\text{but } MR^2 = \frac{b'^2}{a'^2} x^2,$$

$$\text{and } MP^2 = \frac{b'^2}{a'^2} (x^2 - a'^2);$$

$$\begin{aligned} \therefore MR^2 - MP^2 &= \frac{b'^2}{a'^2} \{x^2 - x^2 + a'^2\} \\ &= b'^2; \end{aligned}$$

$$\therefore PR \cdot Pr = b'^2.$$

Let Rr be perpendicular to AX , fig. p. 206, then

$$PR \cdot Pr = b'^2.$$

303. *To find the equation to the hyperbola, when referred to its asymptotes.*

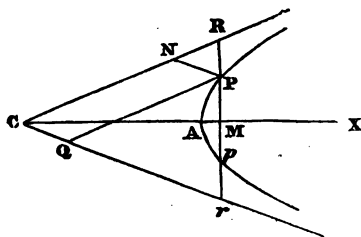
Let P be any point whatever in the hyperbola, draw PN , PQ parallel to the asymptotes, and assume

$$CN = x, \quad NP = y, \quad \angle NCX = \theta.$$

$$\text{Then } \frac{PN}{PR} = \frac{\sin R}{\sin N} = \frac{\cos \theta}{\sin 2\theta} = \frac{1}{2 \sin \theta};$$

$$\therefore y = \frac{PR}{2 \sin \theta}.$$

$$\text{In like manner, } x = \frac{Pr}{2 \sin \theta};$$



$$\therefore xy = \frac{PR \cdot Pr}{4 \sin^2 \theta} = \frac{1}{4} \frac{b^2}{\sin^2 \theta} \dots\dots(1).$$

$$\text{But } \tan^2 \theta = \frac{b^2}{a^2} = \frac{\sin^2 \theta}{1 - \sin^2 \theta};$$

$$\therefore \frac{1}{\sin^2 \theta} = \frac{a^2 + b^2}{b^2};$$

therefore substituting in (1),

$$xy = \frac{1}{4} (a^2 + b^2),$$

or, denoting this constant quantity by m^2 ,

$$xy = m^2,$$

which is the equation required.

304. *Having given the equation to the hyperbola in terms of its axes, to find the equation when it is referred to the asymptotes.*

$$\text{Let } CM = x', \quad MP = y'.$$

Then by the last Article,

$$PR = 2y \sin \theta,$$

$$\text{and } Pr = 2x \sin \theta;$$

$$\therefore Pr - PR, \text{ or } 2y' = 2(x - y) \sin \theta,$$

$$\therefore y' = (x - y) \sin \theta.$$

$$\text{In like manner, } x' = (x + y) \sin \theta.$$

Substituting these values in the equation

$$\frac{y'^2}{b^2} - \frac{x'^2}{a^2} = -1,$$

and observing that

$$\sin^2 \theta = \frac{b^2}{a^2 + b^2},$$

$$\text{we have } \frac{(x - y)^2}{a^2 + b^2} - \frac{(x + y)^2}{a^2 + b^2} = -1;$$

$$\therefore 4xy = a^2 + b^2,$$

$$\text{or } xy = \frac{1}{4}(a^2 + b^2),$$

which is the equation required.

305. *The asymptotes being assumed as axes, to find the equation in the tangent at a given point (x' , y').*

Any other point (x'' , y'') being taken in the hyperbola near the first, the equation to a straight line drawn through (x' , y') and (x'' , y'') is

$$y - y' = \frac{y'' - y'}{x'' - x'} (x - x') \dots \dots \dots (1);$$

but because these points are in the hyperbola, we have

$$x' y' = m^2,$$

$$\text{and } x'' y'' = m^2;$$

$$\therefore x''y'' - x'y' = 0;$$

$$\therefore y'' = \frac{x'y'}{x''};$$

$$\begin{aligned}\therefore y'' - y' &= \frac{x'y'}{x''} - y' \\ &= -\frac{y'}{x''}(x'' - x');\end{aligned}$$

$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{y'}{x''},$$

therefore by substitution in (1) the equation to the secant becomes

$$y - y' = -\frac{y'}{x''}(x - x').$$

Let the point (x'', y'') be now supposed to coincide with (x', y') , then $x'' = x'$, $y'' = y'$, and the secant becomes a tangent; therefore the equation to the tangent is

$$y - y' = -\frac{y'}{x'}(x - x').$$

306. In order to simplify this equation, let each side be multiplied by x' ;

$$\therefore yx' - x'y' = -xy' + y'x';$$

$$\begin{aligned}\therefore yx' + xy' &= 2x'y' \\ &= 2m^2.\end{aligned}$$

307. *To find the intersection of the tangent with the asymptotes.*

The equation to the tangent being

$$yx' + xy' = 2m^2,$$

Let the tangent cut the axis of x , as at T ,

$$\text{then } y = 0, \text{ and } x \text{ or } CT = \frac{2m^2}{y};$$

and when it cuts the axis of y , as at t ,

$$\text{then } x = 0, \text{ and } y \text{ or } Ct = \frac{2m^2}{x'}.$$

$$308. \text{ COR. Hence } CT \cdot Ct = \frac{4m^4}{x'y'} = \frac{4m^4}{m^2} = 4m^2;$$

$$\therefore \frac{1}{2} CT \cdot Ct \times \sin TCt = 2m^2 \sin TCt,$$

that is,

$$\text{the area of the triangle } TCt = 2m^2 \sin TCt.$$

In other words, *if the tangent at any point be produced to meet the asymptotes, the area of the triangle so cut off will be constant.*

309. *Having given one point in the hyperbola, and the position of the asymptotes, to find the direction and magnitude of the transverse and conjugate diameters.*

Let C be the centre, (Fig. p. 203,) CX' CY' the asymptotes, and P the given point in the hyperbola.

(1) To find the direction of the diameters.

Bisect the angle $X'CY'$ and its supplement, by the lines CX and CY ; then CX , CY will evidently be the direction of the transverse and conjugate diameters, respectively.

(2) To find their magnitude.

If the co-ordinates of the given point P be x' , y' , we have

$$x'y' = \frac{a^2 + b^2}{4};$$

$$\therefore a^2 + b^2 = 4x'y' \dots\dots\dots (1),$$

$$\text{but } \pm \frac{b}{a} = \tan \frac{1}{2} X'CY' = \tan \theta;$$

$$\therefore \frac{b^2}{a^2} = \frac{\sin^2 \theta}{\cos^2 \theta},$$

or $b^2 \cos^2 \theta = a^2 \sin^2 \theta$, therefore substituting for b^2 its value in (1),

$$(4x'y' - a^2) \cos^2 \theta = a^2 \sin^2 \theta;$$

$$\begin{aligned} \therefore 4x'y' \cos^2 \theta &= a^2 (\sin^2 \theta + \cos^2 \theta) \\ &= a^2; \end{aligned}$$

$$\therefore a = \pm 2 \cos \theta \sqrt{x'y'},$$

$$\text{and } b = \pm 2 \sin \theta \sqrt{x'y'};$$

therefore the magnitude of the transverse and conjugate diameters is found.

ON THE CONIC SECTIONS IN GENERAL.

CHAPTER I.

ON THE GENERAL EQUATION TO THE THREE CURVES.

310. *AN arc of a Conic Section being traced upon a plane, to determine to which of the three curves it belongs.*

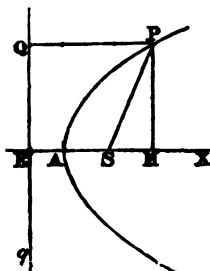
Draw a straight line bisecting any two parallel chords, and also a second straight line bisecting any other two parallel chords. Then according as the point of intersection of these straight lines, that is, the centre, is on the concave or the convex side of the given arc, the arc belongs to the ellipse or the hyperbola.

When the straight lines are parallel, the arc belongs to the parabola.

The position of the axis in the parabola, and of the axes and foci in the ellipse and hyperbola, has already been determined.

311. *A Conic Section being defined, Art. 79, to be the locus of a point, whose distances from a fixed point and a straight line given in position are to each other in a constant ratio, it is required to find its equation.*

Let S be the fixed point or *focus*, EQ the straight line given in position, or *directrix*, P any point in the curve: through S draw the indefinite straight line ESX



perpendicular to the directrix, from P let fall the perpendiculars PM , PQ on AX , EQ , and join P , S .

Let the given ratio of $PS : PQ$ be as $e : 1$, then if SE be divided in A , so that $AS : AE :: e : 1$, A is a point in the curve.

If $AM = x$, $MP = y$ and $AS = m'$, it may be shewn as in Arts. 127 and 218, that the equation required is

$$y^2 = 2m'(1 + e)x - (1 - e^2)x^2.$$

312. COR. 1. Let $e = 1$, then

$$y^2 = 4m'x,$$

the equation to the parabola.

Let $e < 1$, then

$$y^2 = 2m'(1 + e)x - (1 - e^2)x^2,$$

the equation to the ellipse.

Let $e > 1$, then

$$y^2 = 2m'(e + 1)x + (e^2 - 1)x^2,$$

the equation to the hyperbola.

313. COR. 2. Let $2a$, $2b$ be the axes in the ellipse and hyperbola.

Then in the ellipse,

$$a = \frac{m'}{1-e}, \quad \frac{b}{a} = \pm \sqrt{1-e^2};$$

$$\therefore b = \pm m' \sqrt{\frac{1-e}{1+e}}.$$

In the hyperbola

$$a = \frac{m'}{e-1}, \quad \frac{b}{a} = \pm \sqrt{e^2-1};$$

$$\therefore b = \pm m' \sqrt{\frac{e-1}{e+1}}.$$

314. We have already demonstrated, when treating separately of each curve, that the equation retains the same form, whether the axes be rectangular or oblique. Hence

(1) The simplest form of the equation to the Conic Sections, *taken collectively*, and referred to any diameter and the tangent at its vertex, is

$$y^2 = mx + nx^2,$$

n being = 0 in the parabola, negative in the ellipse, and positive in the hyperbola.

(2) The simplest form of the equation to the Conic Sections, *taken separately*, is

$$y^2 = Qx$$

in the parabola, referred to conjugate axes:

$$\text{and } My^2 + Nx^2 = P$$

in the ellipse and the hyperbola, referred to the centre and any system of conjugate diameters.

For the equation to the parabola was proved (111) to be

$$y^2 = 4SP \cdot x,$$

which is of the form

$$y^2 = Qx.$$

Also the equation to the ellipse and hyperbola is (171) and (266),

$$a'^2 y^2 \pm b'^2 x^2 = \pm a'^2 b'^2,$$

the upper sign for the ellipse, the lower for the hyperbola, and this is evidently of the form

$$M y^2 + N x^2 = P,$$

$$a' \text{ being } = \sqrt{\frac{P}{N}} \text{ and } b' = \sqrt{\frac{P}{M}}.$$

In the ellipse M , N and P are all positive; in the hyperbola M and N have different signs, and P may be either positive or negative.

315. *To find the polar equation to a Conic Section.*

Let AP be an arc of a Conic Section, of which S is the focus, and EQ the directrix. See the last figure.

Then the same construction and notation remaining,

$$\text{let } SP = r, \quad \angle ASP = \omega.$$

Since $SP = e \cdot PQ$ by the definition;

$$\therefore r = e(AE + AM)$$

$$= e \left(\frac{m'}{e} + a \right)$$

$$= e \left(\frac{m'}{e} + m' - r \cos \omega \right)$$

$$= m' + em' - er \cos \omega;$$

$$\therefore r = \frac{m'(1+e)}{1+e \cos \omega},$$

which is the equation required.

316. COR. 1. If $e = 1$, $r = \frac{2m'}{1 + e \cos \omega}$, which is the polar equation to the parabola.

$$\text{If } e \text{ is } < \text{ or } > 1, r = \frac{m'(1 + e)}{1 + e \cos \omega},$$

but $m' = \pm a(1 - e)$ in the ellipse or hyperbola respectively;

$$\therefore r = \frac{\pm a(1 - e^2)}{1 + e \cos \omega},$$

which is the polar equation to the ellipse or hyperbola.

317. From the two equations

$$y^2 = mx + nx^2$$

$$\text{and } r = \frac{m'(1 + e)}{1 + e \cos \omega},$$

the various properties of a *Conic Section in general* may be deduced, in precisely the same manner in which they were derived from the corresponding equations of each of the three curves in particular.

The following problem will illustrate the application of the equation.

$$y^2 = mx + nx^2.$$

In the plane of a Conic Section to determine a fixed point (x', y') , such that its distance from any point (x, y) of the curve may be a rational function of the abscissa.

If r denote the distance between the points (x', y') and (x, y) , we have (36)

$$r = \sqrt{\{(x - x')^2 + (y - y')^2\}};$$

therefore substituting for y its value,

$$\begin{aligned} r &= \sqrt{\{(x - x')^2 + (\sqrt{mx + nx^2} - y')^2\}} \\ &= \sqrt{x^2 - 2xx' + x'^2 + mx + nx^2 + y'^2 - 2y'\sqrt{mx + nx^2}}. \end{aligned}$$

but in order that r may be a rational function of x ,

$$2y'\sqrt{mx + x^2} \text{ must } = 0;$$

$$\therefore y' = 0,$$

that is, the point required must be situated on the axis of x .

Hence

$$\begin{aligned} r &= \sqrt{\{x^2 - 2xx' + x'^2 + mx + nx^2\}} \\ &= \sqrt{\{x^2(1+n) + (m-2x')x + x'^2\}}. \end{aligned}$$

The quantity under the radical must evidently be a complete square, in order that the condition of the problem may be fulfilled;

$$\therefore 4x^2x'^2(1+n) = (m-2x')^2x^2;$$

$$\therefore 2x'\sqrt{1+n} = m-2x';$$

$$\therefore x' = \frac{\frac{1}{2}m}{1 + \sqrt{1+n}},$$

which is the distance of the point required from the vertex.

In the parabola

$$n = 0; \therefore x' = \frac{1}{4}m.$$

In the ellipse and hyperbola

$$m = 2m'(1+e) \text{ and } n = \mp(e^2-1); \text{ Art. 311.}$$

$$\therefore x' = m'.$$

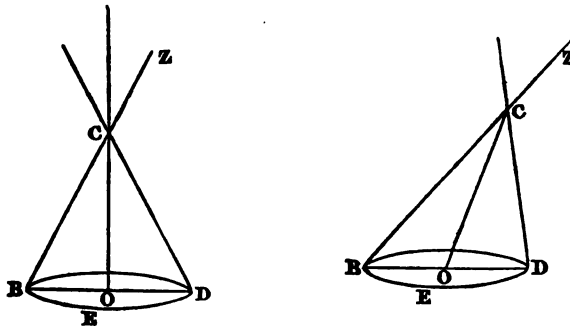
It appears therefore in each case, that the point required is the focus.

CHAPTER II.

ON THE SECTIONS OF THE CONE.

318 DEF. Let C be a fixed point above the plane of a given circle BED , and BCZ an indefinite straight line which always passes through C , whilst its extremity B moves over the circumference BED ; then BCZ will describe by its revolution a solid figure called a *cone*.

The point C is called the *vertex*, the circle BED the *base*, and the line CO , which joins the vertex with the centre of the base, the *axis* of the cone.



The cone is denominated a *right*, or an *oblique*, cone, according as the axis is at right angles, or inclined, to the plane of the base.

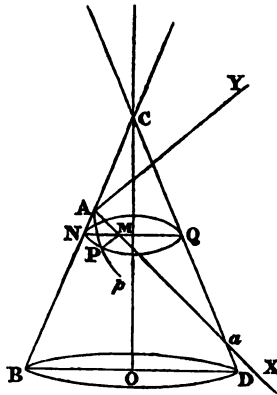
The surface of a cone is composed of two similar portions, one above, and the other below, the vertex; each of these portions is called a *sheet**.

* *Sheet* is to a *surface*, what *branch* is to a *curve*.

It is evident from the manner in which a cone is generated, that every section of it, made by a plane parallel to the base, is a circle.

319. *To find the nature of the curve, which results from the intersection of a right cone by a plane.*

Let APp be the curve formed by the intersection of a right cone by a plane; through the axis CO draw a plane BCD perpendicular to the given plane, then their intersection will be the straight line Aa . In Aa take any point M , through which draw a plane parallel to the base, then its intersections with the cone and the given plane will be, respectively, the circle NPQ and the straight line MP , which being perpendicular to Aa and NQ , will be a common ordinate to both curves.



Assume Aa as the axis of x , and AY , at right angles to Aa , as the axis of y , and let $AM = x$, $MP = y$; also take $AC = \delta$, angle $BCD = \alpha$, and angle $CAa = \theta$.

$$\text{Then } \frac{Aa}{AC} = \frac{\sin ACa}{\sin AaC} = \frac{\sin \alpha}{\sin (\alpha + \theta)};$$

$$\therefore Aa = \frac{\sin \alpha}{\sin (\alpha + \theta)} \delta;$$

$$\therefore Ma = Aa - x = \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x.$$

Now, since P is a point in the circle NPQ ,

$$MP^2 \text{ or } y^2 = NM \cdot MQ \dots \dots \dots (1);$$

$$\text{but } NM = MA \frac{\sin NAM}{\sin ANM} = x \frac{\sin CAa}{\cos NCM}$$

$$= \therefore \frac{x \sin \theta}{\cos \frac{1}{2} a},$$

$$\text{and } MQ = Ma \frac{\sin AaC}{\sin MQa} = Ma \frac{\sin (a + \theta)}{\sin NQC}$$

$$= \therefore \left\{ \frac{\delta \sin a}{\sin (a + \theta)} - x \right\} \frac{\sin (a + \theta)}{\cos \frac{1}{2} a};$$

therefore by substitution in (1),

$$\begin{aligned} y^2 &= \frac{x \sin \theta}{\cos \frac{1}{2} a} \cdot \frac{\sin (a + \theta)}{\cos \frac{1}{2} a} \left\{ \frac{\delta \sin a}{\sin (a + \theta)} - x \right\} \\ &= \frac{\sin \theta}{\cos^2 \frac{1}{2} a} \{ \delta \sin a \cdot x - \sin (a + \theta) x^2 \}, \end{aligned}$$

which (311) is the equation to a conic section.

320. *When the intersecting plane is parallel to the generating line CD , the section is a parabola.*

The plane being parallel to CD , we have in the above equation,

$$a + \theta = \pi, \text{ therefore } \sin (a + \theta) = \sin \pi = 0;$$

$$\text{also } \sin \theta = \sin (\pi - a) = \sin a,$$

therefore the equation becomes

$$y^2 = \frac{\delta \sin^2 a}{\cos^2 \frac{1}{2} a} \cdot x = \frac{4 \delta \sin^2 \frac{1}{2} a \cos^2 \frac{1}{2} a}{\cos^2 \frac{1}{2} a} x = 4 \delta \sin^2 \frac{a}{2} x,$$

which is the equation to a parabola, whose *latus rectum*

$$= 4 \delta \sin^2 \frac{a}{2}.$$

321. **COR.** If the plane pass through the vertex of the cone, then $\delta = 0$, and the equation to the section becomes $y^2 = 0$, which is the equation to the straight line CD .

322. *When the intersecting plane meets only one sheet of the cone, the section is an ellipse.*

Let the plane meet CB and CD , then $\alpha + \theta$ being $< \pi$, $\sin(\alpha + \theta)$ is positive; therefore the equation to the section is

$$y^2 = \frac{\sin \theta}{\cos^2 \frac{1}{2} \alpha} \{ \delta \sin \alpha \cdot x - \sin(\alpha + \theta) x^2 \},$$

which is the equation to the *ellipse*.

When the intersecting plane is parallel to the base, the ellipse becomes a circle.

323. **COR. 1.** To find the *latus rectum* and *axes*.

Comparing the above equation with

$$y^2 = \frac{b^2}{a^2} \{ 2ax - x^2 \},$$

$$\text{or with } y^2 = \frac{2b^2}{a} x - \frac{b^2}{a^2} x^2,$$

$$\text{we have } \frac{2b^2}{a}, \text{ or the latus rectum} = \frac{\delta \sin \alpha \sin \theta}{\cos^2 \frac{1}{2} \alpha}$$

$$= 2\delta \tan \frac{\alpha}{2} \sin \theta \dots \dots \dots (1),$$

$$\text{also } \frac{b^2}{a^2} = \frac{\sin \theta \cdot \sin(\alpha + \theta)}{\cos^2 \frac{1}{2} \alpha} \dots \dots \dots (2),$$

$$\therefore a = \frac{\delta \sin \alpha}{2 \sin(\alpha + \theta)};$$

$$\begin{aligned}
 \text{whence } b^2 &= \frac{a}{2} \cdot 2\delta \tan \frac{a}{2} \sin \theta \\
 &= \frac{\delta \sin a}{2 \sin(a + \theta)} \cdot \delta \tan \frac{a}{2} \sin \theta \\
 &= \delta^2 \sin^2 \frac{a}{2} \cdot \frac{\sin \theta}{\sin(a + \theta)}; \\
 \therefore b &= \delta \sin \frac{a}{2} \sqrt{\frac{\sin \theta}{\sin(a + \theta)}}.
 \end{aligned}$$

324. COR. 2. If the plane pass through the vertex, then $\delta = 0$, and the equation becomes

$$y^2 = - \frac{\sin(a + \theta) \sin \theta}{\cos^2 \frac{1}{2} a} x^2,$$

which, since it can be satisfied only by supposing $x = 0$, $y = 0$, is the equation to the point C .

325. *When the intersecting plane meets both sheets of the cone, the section is an hyperbola.*

In this case, $a + \theta > \pi$, and because $\sin(a + \theta)$ is negative, the equation to the section is

$$y^2 = \frac{\sin \theta}{\cos^2 \frac{1}{2} a} \{ \delta \sin a \cdot x + \sin(a + \theta) x^2 \},$$

which is the equation to the *hyperbola*.

The *latus rectum* and *axes* of the hyperbola may be determined in the same manner as in the ellipse.

326. COR. If the plane pass through the vertex, then $\delta = 0$, and the equation becomes

$$\begin{aligned}
 y^2 &= \frac{\sin \theta}{\cos^2 \frac{1}{2} a} \sin(a + \theta) x^2; \\
 \therefore y &= \pm \frac{\sqrt{\sin \theta \sin(a + \theta)}}{\cos \frac{1}{2} a} x,
 \end{aligned}$$

which are the equations to CB , CD ; hence the section becomes in this case the two generating lines of the cone.

It appears, therefore, from the preceding investigation, that if a right cone be cut by a plane, the section will be

- (1) A *parabola*, when the plane is parallel to the generating line.

The particular case is a straight line.

- (2) An *ellipse*, when the plane meets only one sheet of the cone.

The particular cases are a point and a circle.

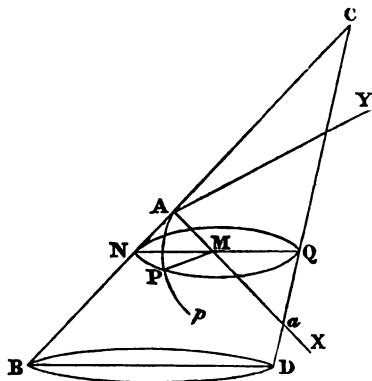
- (3) An *hyperbola*, when the plane meets both sheets of the cone.

The particular case is two straight lines which intersect each other.

327. To find the nature of the curve which results from the intersection of an oblique cone by a plane.

Let APp be the curve formed by the section of an oblique cone by a plane.

The construction is the same as before, excepting that the



line MP is no longer perpendicular both to Aa and NQ , but

only to the latter; we shall therefore assume as oblique axes the lines Aa , and AY , the latter being parallel to MP .

$$\text{Hence, as before } Aa = \frac{\delta \sin \alpha}{\sin (\alpha + \theta)},$$

$$\text{and } Ma = \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x \dots \dots \dots (1);$$

$$\text{also } y^2 = NM \cdot MQ;$$

$$\text{now } NM = x \frac{\sin \theta}{\sin B},$$

$$\begin{aligned} \text{and } MQ &= Ma \frac{\sin (\alpha + \theta)}{\sin (\alpha + B)}, \\ &= \frac{\sin (\alpha + \theta)}{\sin (\alpha + B)} \left\{ \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} - x \right\}; \end{aligned}$$

$$\therefore y^2 = \frac{\sin \theta}{\sin B \sin (\alpha + B)} \{ \delta \sin \alpha \cdot x - \sin (\alpha + \theta) x^2 \},$$

which, according as the given plane is parallel to CD , or meets one sheet, or both sheets, of the cone, is the equation to a parabola, ellipse, or hyperbola, referred to oblique axes.

328. *To find in what cases the section of an oblique cone by a plane is a circle.*

Having put the equation in the last Art. under the form

$$y^2 = \frac{\sin \theta \sin (\alpha + \theta)}{\sin B \sin (\alpha + B)} \left\{ \frac{\delta \sin \alpha}{\sin (\alpha + \theta)} x - x^2 \right\},$$

it is evident that the section will be a circle, when the coefficient

$$\frac{\sin \theta \sin (\alpha + \theta)}{\sin B \sin (\alpha + B)} = 1,$$

$$\text{or } \sin \theta \sin (\alpha + \theta) = \sin B \sin (\alpha + B),$$

$$\text{or } \cos \alpha - \cos (\alpha + 2\theta) = \cos \alpha - \cos (\alpha + 2B),$$

$$\therefore \cos (\alpha + 2\theta) \text{ must } = \cos (\alpha + 2B),$$

$$\therefore \alpha + 2\theta = \alpha + 2B \dots \dots \dots (1),$$

$$\text{or } = 2\pi - (\alpha + 2B) \dots \dots \dots (2)$$

$$\text{First, if } \alpha + 2\theta = \alpha + 2B,$$

$$\text{then } \theta = B,$$

that is, *when the plane is parallel to the base of the cone, the section is a circle.*

$$\text{Secondly, if } \alpha + 2\theta = 2\pi - (\alpha + 2B),$$

$$\text{then } 2\alpha + 2\theta + 2B = 2\pi,$$

$$\text{or } \alpha + \theta + B = \pi = \therefore \alpha + D + B;$$

$$\therefore \theta = D,$$

hence, *when the angle CAX is = CDB, the section is also a circle.* The latter section is denominated the *subcontrary section* of the cone.

ON LINES OF THE SECOND ORDER.

CHAPTER I.

ON THE LOCUS OF THE GENERAL EQUATION OF THE SECOND DEGREE.

SECTION I.

ON THE CONSTRUCTION AND NATURE OF THE LOCUS IN GENERAL.

DEF. THE locus of the general equation of the second degree between two variables is called a *Line of the Second Order*.

329. *The general equation of the second degree being*

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

it is required to construct the curve of which it is the locus.

Resolving the equation in terms of x and y successively, we have

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a} \sqrt{\{(b^2-4ac)x^2 - 2(2ae-bd)x + d^2-4af\}},$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c} \sqrt{\{(b^2-4ac)y^2 - 2(2cd-be)y + e^2-4cf\}},$$

Let the known coefficients in the rational parts of these two equations be represented by single letters, M ; N , N' ; P , P' ; and let

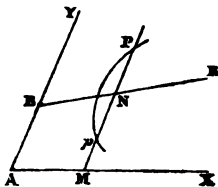
$$-\frac{b}{2a} = m, \quad -\frac{d}{2a} = n; \quad -\frac{b}{2c} = m', \quad -\frac{e}{2c} = n';$$

$$\text{then } y = mx + n \pm \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}} \dots\dots\dots (1),$$

$$x = m'y + n' \pm \frac{1}{2c} \sqrt{\{My^2 - 2N'y + P'\}} \dots\dots\dots (2).$$

(1) To construct the values of y .

The required locus being referred to any axes whatever



AX , AY , it is evident that the rational part of (1), namely $mx + n$, is the equation to a straight line. Let this line, constructed by Art. 14, be BR .

In AX take any point M , and draw MP parallel to AY , meeting BR in N .

Then if $AM = x$, MN will $= mx + n$.

Again, take NP , Np in opposite directions each equal to the irrational part of equation (1), then

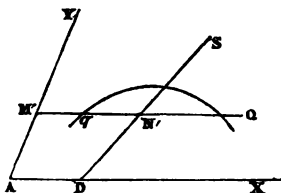
$$MP = mx + n + \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}},$$

$$Mp = mx + n - \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}},$$

whence P , p are two points in the required locus; and for every other assumed value of x , which shall render the irrational part of (1) real, two other points may be determined.

When the assumed value of x renders the irrational part of (1) imaginary, there is no corresponding point in the curve; and when it renders it = 0, the curve is then reduced to a point.

(2) To construct the values of x .



Let DS be the straight line, of which the rational part of (2) is the equation.

In AY take any point M' , and draw $M'Q$ parallel to AX , meeting DS in N' .

Then if $AM' = y$, $M'N'$ will = $m'y + n'$.

Also, if $N'Q$, $N'q$ be taken, in opposite directions, equal to the irrational part of (2), the points Q , q , as before, belong to the locus required.

330. The straight lines BR , DS bisect all chords which are parallel to AY , AX respectively.

Each of the lines BR , DS , therefore, is a *diameter*. See Def. 1, Art. 107.

331. To find the point in which the diameters BR , DS intersect each other.

The equations to these diameters, if m , n and m' , n' be replaced by their values, Art. 329, are

$$y = -\frac{b}{2a}x - \frac{d}{2a},$$

$$x = -\frac{b}{2c}y - \frac{e}{2c}.$$

When the diameters intersect, the values of x and y in their equations are identical;

therefore eliminating y we have

$$-\frac{b}{2a}x - \frac{d}{2a} = -\frac{2c}{b}x - \frac{e}{b};$$

$$\therefore x = -\frac{bd - 2ae}{b^2 - 4ac},$$

$$\text{and } y = -\frac{be - 2cd}{b^2 - 4ac},$$

which are the co-ordinates of the point of intersection required.

This point is *the centre*.

332. The nature of the curves, which are the loci of the general equation of the second degree, will evidently depend upon the irrational part of the two equations in Art. 329.

Now such values may be assigned to x in the one, and to y in the other, as shall render the first term in each of the trinomials,

$$Mx^2 - 2Nx + P, \text{ and } My^2 - 2N'y + P',$$

greater than the two others taken together.

The sign therefore of these two trinomials will depend upon that of the first term,

$$Mx^2 \text{ or } My^2,$$

that is, upon the sign of M , since x^2 and y^2 are necessarily positive.

As the nature of the curve has thus been shewn to depend upon the value of M , and as M may be equal to,

less or greater than, nothing, it follows that the locus of the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

comprehends three distinct curves, corresponding to these three cases;

(1) *The Parabola*, when $b^2 - 4ac = 0$.

(2) *The Ellipse*, when $b^2 - 4ac < 0$.

(3) *The Hyperbola*, when $b^2 - 4ac > 0$.

333. It was shewn (331), that the centre is a point whose co-ordinates are

$$x = -\frac{bd - 2ae}{b^2 - 4ac}, \quad y = -\frac{be - 2cd}{b^2 - 4ac}.$$

Since these co-ordinates are infinitely great, when

$$b^2 - 4ac = 0,$$

it follows that the centre is at an infinite distance, or in other words, that the parabola has no centre.

The ellipse and hyperbola are frequently, in contradistinction to the parabola, called *Central Curves*.

ON THE PARABOLA.

In this case $y = mx + n \pm \frac{1}{2a} \sqrt{-2Nx + P}$;

If $AD = a$, then $y = mx + n \pm \frac{1}{2a} \sqrt{-2N(x-a)}$.

* In this and the two following sections the axes may have any inclination whatever; for the sake of simplicity they are assumed rectangular.

When $x > a$, $-2N(x-a)$ is positive, and for each assumed value of x there are two equal values of y , with opposite signs.

How great soever x be taken, the values of y remain real; hence the curve consists of two infinite branches dP , dp extending to the right of d .

(2) Let N be positive,

Then if $x > a$, $-2N(x-a)$ is negative, and if $x < a$, $-2N(x-a)$ is positive, the curve therefore in this case extends infinitely to the *left* of d , and is limited towards the right by the line Dd .

The curve thus determined is evidently the *Parabola*.

(3) Let $N = 0$:

$$\text{then } y = mx + n \pm \frac{1}{2a} \sqrt{P},$$

which is the equation to *two parallel straight lines*.

In the axis of y take BD , BD' , in opposite directions from B , each $= \frac{1}{2a} \sqrt{P}$, and through D , D' draw DS , DS' parallel to BR , then these will be the lines in question.

If now $P = 0$, then $y = mx + n$, which is the equation to BR .

If P be negative, the values of y are imaginary.

334. COR. 1. Since $b^2 - 4ac = 0$;

$$\therefore b = \pm 2 \sqrt{ac},$$

therefore the general equation becomes by substitution

$$ay^2 \pm 2 \sqrt{ac} \cdot xy + cx^2 + dy + ex + f = 0;$$

$$\text{or } (a^{\frac{1}{2}}y \pm c^{\frac{1}{2}}x)^2 + dy + ex + f = 0.$$

Hence when the first three terms form a complete square, the equation represents a parabola.

335. COR. 2. Since $b^2 - 4ac = 0$,

if $b = 0$, then a or c must $= 0$,

and conversely,

if a or $c = 0$, then b must $= 0$.

The equation to the parabola may therefore occur under either of the two following forms:

$$ay^2 + dy + ex + f = 0,$$

$$\text{or } cx^2 + dy + ex + f = 0.$$

Observation. The varieties of the parabola are

(1) *Two parallel straight lines:*

when N , or $bd - 2ae = 0$.

(2) *One straight line:*

when N and P , or $bd - 2ae$, and $d^2 - 4af$ each $= 0$.

(3) *An imaginary line:*

when N , or $bd - 2ae = 0$, and $d^2 - 4af < 0$.

336. *To determine the form of the general equation to the parabola, when it represents any one of its varieties.*

When $M = 0$,

$$y = mx + n \pm \frac{1}{2a} \sqrt{-2Nx + P}$$

$$= -\frac{bx + d}{2a} \pm \frac{1}{2a} \sqrt{-2Nx + P}.$$

In all the varieties, as has been just shewn, $N = 0$;

$$\therefore y = -\frac{bx + d}{2a} \pm \frac{1}{2a} \sqrt{P},$$

$$\text{or, } \{2ay + bx + d\}^2 - P = 0 \dots \dots \dots (1).$$

Here P may be positive, or negative, or $= 0$.

(1) Let P be positive,

then equation (1) is evidently the product of the two factors

$$\{2ay + bx + d\} + \sqrt{P}, \text{ and } \{2ay + bx + d\} - \sqrt{P};$$

each of which being supposed $= 0$, we have

$$y = -\frac{b}{2a}x - \frac{d + \sqrt{P}}{2a},$$

and

$$y = -\frac{b}{2a}x - \frac{d - \sqrt{P}}{2a},$$

which are the equations to two parallel straight lines.

(2) Let $P = 0$,

$$\text{then } (2ay + bx + d)^2 = 0;$$

$$\therefore y = -\frac{b}{2a}x - \frac{d}{2a},$$

which is the equation to one straight line.

(3) Let P be negative,

$$\text{then } \{2ay + bx + d\}^2 + P = 0,$$

which, since P is independent of x and y , cannot $= 0$; therefore the curve is an imaginary line.

Hence it appears that the general equation to the parabola represents

(1) *Two parallel straight lines:*

when it is the difference of two squares, one of which is a constant quantity.

(2) *One straight line:*

when it is a complete square.

(3) *An imaginary line:*

when it is the sum of two squares, one of which is constant.

SECTION III.

ON THE ELLIPSE.

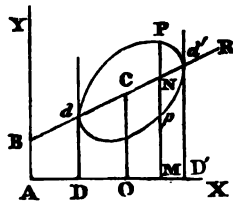
337. *To determine the curve when M, or $b^2 - 4ac < 0$.*

Here $y = mx + n \pm \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}}$;

when the curve intersects the diameter, we have, as in the last case,

$$Mx^2 - 2Nx + P = 0;$$

$$\therefore x = \frac{N}{M} \pm \frac{1}{M} \sqrt{(N^2 - MP)} \dots \dots \dots (1)$$



In AX take $AO = \frac{N}{M}$, and OD, OD' , on opposite sides of O , each $= \frac{1}{M} \sqrt{(N^2 - MP)}$, through D, D' draw $Dd, D'd'$ parallel to AY , meeting BR in d, d' : then d, d' will be the points in which the diameter BR intersects the curve.

Let a, a' be the roots of equation (1), in other words, let

$$AD = a, AD' = a';$$

$$y = mx + n \pm \frac{1}{2a} \sqrt{M(x-a)(x-a')}.$$

These roots may be either real or imaginary.

(1) Let them be real.

If $x < a$, it must necessarily $< a'$, $\therefore x - a$ and $x - a'$ being each negative, their product is positive, and

$$\therefore M(x-a)(x-a')$$

is negative.

Hence the values of y are imaginary, and no part of the curve can lie between A and D .

If $x > a'$, it must also $> a$, and $x - a, x - a'$ being each positive, $M(x-a)(x-a')$ is negative, and therefore no part of the curve can lie beyond D' .

If $x > a$ and $< a'$, then $(x-a)(x-a')$ being negative, $M(x-a)(x-a')$ is positive, and hence for each assumed value of x , intermediate between a , and a' , there are two real values of y , equal to each other, and with opposite signs.

The curve therefore is limited in the direction AX , by the two lines $Dd, D'd'$ parallel to AY .

In a similar manner it may be shewn that the curve is limited in the direction AY , by two lines parallel to AX .

The curve thus determined is evidently *the Ellipse*.

(2) Let the roots be imaginary, and suppose them to be

$$a + \beta\sqrt{-1} \text{ and } a - \beta\sqrt{-1};$$

then

$$\{x - (a + \beta\sqrt{-1})\} \{x - (a - \beta\sqrt{-1})\} = (x-a)^2 + \beta^2,$$

which is necessarily a positive quantity;

$$\therefore M \{(x-a)^2 + \beta^2\}$$

being negative, the values of y are imaginary.

The curve therefore in this case becomes an imaginary line.

338. COR. 1. If the roots a, a' be real and equal,

$$\begin{aligned} \text{then } y &= mx + n \pm \frac{1}{2a} \sqrt{-M(x-a)^2} \\ &= mx + n \pm \frac{x-a}{2a} \sqrt{-M}, \end{aligned}$$

which is imaginary, except when $x = a$, in which case

$$y = ma + n.$$

Hence the ellipse is reduced to a point, whose co-ordinates are

$$x = a, \quad y = ma + n.$$

339. COR. 2. Since b^2 is necessarily a positive quantity, it follows that in the equation to the ellipse neither a nor c can = 0, and that they must always have the same sign.

340. COR. 3. The general equation to the circle is (50)

$$(y - y')^2 + 2(y - y')(x - x') \cos \gamma + (x - x')^2 = r^2;$$

therefore, developing, and arranging the result according to the powers of y ,

$$\begin{aligned} y^2 + 2 \cos \gamma \cdot xy + x^2 - 2(y' + x' \cos \gamma)y - 2(x' + y' \cos \gamma)x \\ + y'^2 + x'^2 + 2x'y' \cos \gamma - r^2 = 0. \end{aligned}$$

Comparing this with the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

$$\text{or } y^2 + \frac{b}{a}xy + \frac{c}{a}x^2 + \&c. = 0,$$

we have $2 \cos \gamma = \frac{b}{a}$, and $1 = \frac{c}{a}$; $\therefore b = 2a \cos \gamma$, and $a = c$,
 $\therefore b^2 - 4ac = 4a^2 \cos^2 \gamma - 4a^2 = -4a^2 (1 - \cos^2 \gamma) = -4a^2 \sin^2 \gamma$;
 which is necessarily negative.

Hence the circle is a species of the ellipse, and it is the locus of the equation

$$ay^2 + 2a \cos \gamma \cdot xy + ax^2 + dy + ex + f = 0.$$

When the axes are rectangular,

$$\cos \gamma = 0;$$

therefore the equation becomes

$$y^2 + x^2 + \frac{d}{a}y + \frac{e}{a}x + \frac{f}{a} = 0,$$

which is of the form

$$y^2 + x^2 + Dy + Ex + F = 0.$$

Observation. The varieties of the ellipse are,

(1) *An imaginary line:*

when $N^2 < MP$, or $(bd - 2ae)^2 < (b^2 - 4ac)(d^2 - 4af)$.

(2) *A point:*

when $N^2 = MP$, or $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$.

(3) *A circle:*

when $a = c$ and $\cos x, y = \frac{b}{2a}$.

341. *To determine the form of the general equation to the ellipse, when it represents any one of its varieties.*

(1) Let the roots be imaginary,

$$\text{then } y = -\frac{bx + d}{2a} \pm \frac{1}{2a} \sqrt{M \{ (x - \alpha)^2 + \beta^2 \}};$$

$$\therefore \{2ay + bx + d\}^2 - M(x - \alpha)^2 - M\beta^2 = 0,$$

which, M being negative, is the sum of three squares; the last of which is independent of x and y ; therefore the equation cannot = 0, and the locus is an imaginary line.

(2) Let the roots be equal;

$$\text{then } y = -\frac{bx+d}{2a} \pm \frac{x-a}{2a} \sqrt{M};$$

$$\therefore \{2ay + bx + d\}^2 - M(x-a)^2 = 0,$$

which, since M is negative, is the sum of two positive quantities.

This sum cannot, therefore, = 0, unless each of its parts = 0;

$$\therefore 2ay + bx + d = 0 \text{ and } x - a = 0;$$

$$\therefore x = a \text{ and } y = -\frac{ba + 2d}{2a},$$

which is the equation to a point.

Hence it appears that the general equation to the ellipse represents

(1) *An imaginary line:*

when it is the sum of three squares, the last of which is a constant quantity.

(2) *A point:*

when it is the sum of two positive quantities, each of which is a function of x , y .

(3) *A circle:*

when $a = c$ and $b = 2a \cos \alpha$, y ,
the axes being oblique;

$$\text{or } a = c \text{ and } b = 0,$$

the axes being rectangular.

SECTION IV.

ON THE HYPERBOLA.

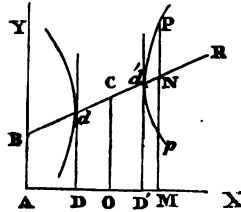
342. *To determine the curve when M , or $b^2 - 4ac > 0$,*

$$\text{Here } y = mx + n \pm \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}}.$$

When the curve intersects the diameter, we have as in the last two cases,

$$\therefore Mx^2 - 2Nx + P = 0;$$

$$\therefore x = \frac{N}{M} \pm \frac{1}{M} \sqrt{(N^2 - MP)}.$$



Having as in the last case drawn the ordinates Dd , $D'd'$, and assumed

$$AD = a, \quad AD' = a',$$

we have

$$y = mx + n \pm \frac{1}{2a} \sqrt{M(x - a)(x - a')}.$$

The roots a , a' may be either real or imaginary.

(1) Let them be real.

If $x < \alpha$, it must also $< \alpha'$;

$$\therefore x - \alpha, \text{ and } x - \alpha'$$

being each negative, their product is positive; and therefore

$$M(x - \alpha)(x - \alpha')$$

is positive;

therefore the values of y are real.

When x is negative, the values of y are still real, because the product of the factors being

$$(x + \alpha)(x + \alpha'),$$

it follows that

$$M(x + \alpha)(x + \alpha')$$

must be positive.

How great soever the negative values of x be taken, there are always two real corresponding values of y : hence the curve extends indefinitely to the *left* of D .

If $x > \alpha$ but $< \alpha'$, that is, if x have any intermediate value between AD and AD' , then $x - \alpha$ being positive, and $x - \alpha'$ negative, their product, and consequently

$$M(x - \alpha)(x - \alpha')$$

is negative; therefore the values of y are imaginary. Hence no part of the curve lies between D and D' :

If $x > \alpha'$, then it is necessarily $> \alpha$; therefore

$$x - \alpha \text{ and } x - \alpha'$$

being each positive,

$$M(x - \alpha)(x - \alpha')$$

is so likewise, and therefore the values of y are real.

How great soever x be taken, the quantity

$$M(x-a)(x-a')$$

still continues positive; therefore the values of y are real, and the curve extends indefinitely to the *right* of D' .

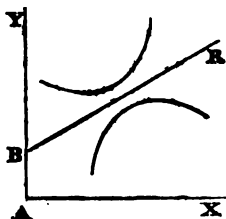
(3) Let the roots be imaginary: then if they be denoted, as in the last case, by

$$a + \beta\sqrt{-1} \text{ and } a - \beta\sqrt{-1},$$

$$M\{x-(a+\beta\sqrt{-1})\} \cdot \{x-(a-\beta\sqrt{-1})\} = M\{(x-a)^2 + \beta^2\},$$

which being always positive, the values of y are real.

Hence the curve consists in this case of two branches, extending indefinitely above and below the diameter BR .



The curve thus determined is evidently *the hyperbola*.

343. COR. 1. If the roots a, a' be real and equal, then

$$y = mx + n \pm \frac{x-a}{2a} \sqrt{M},$$

$$\text{but } a = \frac{N}{M}; \therefore y = mx + n \pm \frac{\sqrt{M}}{2a} \left(x - \frac{N}{M}\right),$$

which values of y are real.

These are the equations to two straight lines which, since the coefficients of x are not the same, intersect each other.

To find where they intersect.

In the above equation let

$$x = a \text{ or } \frac{N}{M},$$

then y , the ordinate of the point of intersection,

$$= ma + n;$$

but this is the ordinate of the centre; therefore the two lines intersect at the centre.

344. COR. 2. Since b^2 is necessarily positive, either a or c , or both of them, may = 0.

If $b = 0$, then a and c must have different signs.

345. COR. 3. The axes being supposed rectangular, let

$$b = 0 \text{ and } a = -c,$$

then the equation becomes

$$ay^2 - ax^2 + dy + ex + f = 0,$$

$$\text{or } a(y^2 - x^2) + dy + ex + f = 0;$$

therefore dividing by a , and transposing,

$$\left(y^2 + \frac{d}{a}y\right) - \left(x^2 - \frac{e}{a}x\right) = -\frac{f}{a};$$

adding $\frac{d^2 - e^2}{4a^2}$ to each, we have

$$\left(y + \frac{d}{2a}\right)^2 - \left(x - \frac{e}{2a}\right)^2 = \frac{d^2 - e^2 - 4af}{4a^2},$$

or if

$$y + \frac{d}{a} = y', \quad x - \frac{e}{a} = x' \quad \text{and} \quad \frac{d^2 - e^2 - 4af}{4a^2} = r^2,$$

$$y'^2 - x'^2 = r^2,$$

which (224) is the equation to the equilateral hyperbola.

Observation.

The varieties of the hyperbola are

(1) *Two straight lines which intersect each other :*

when $N^2 = MP$, or $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$.

(2) *An equilateral hyperbola :*

when, the axes being rectangular,

$$b = 0 \quad \text{and} \quad a = -c.$$

346. *To determine the general form of the equation to the hyperbola, when it represents either of its varieties.*

Let the roots be equal, then as in the case of the ellipse,

$$(2ay + bx + d)^2 - M(x - a)^2 = 0,$$

and, M being positive, this quantity represents the difference of two squares, and is therefore equal to the product of the two factors

$$\{2ay + bx + d\} + \sqrt{M}(x - a),$$

$$\{2ay + bx + d\} - \sqrt{M}(x - a),$$

each of which must = 0;

$$\therefore y = -\frac{b + \sqrt{M}}{2a}x - \frac{d - a\sqrt{M}}{2a},$$

$$\text{and } y = -\frac{b - \sqrt{M}}{2a}x - \frac{d + a\sqrt{M}}{2a},$$

which are the equations to two straight lines that intersect each other.

Hence it appears that the general equation to the hyperbola represents,

(1) *Two straight lines which intersect each other :*

when it is the difference of two squares, each of which is a function of x, y .

And, as was shewn in the last Article,

(2) *An equilateral hyperbola :*

when $b = 0$ and $a = -c$, the axes being rectangular.

347. *From the general equation to the hyperbola to deduce the equation to its asymptotes.*

The general equation is

$$y = mx + n \pm \frac{1}{2a} \sqrt{\{Mx^2 - 2Nx + P\}},$$

in which M is a positive quantity.

Now

$$\{Mx^2 - 2Nx + P\}^{\frac{1}{2}} = M^{\frac{1}{2}}x \left(1 - \frac{2Nx - P}{Mx^2}\right)^{\frac{1}{2}},$$

and the latter, expanded by the binomial theorem, becomes,

$$\begin{aligned} M^{\frac{1}{2}}x \left\{1 - \frac{1}{2} \frac{2Nx - P}{Mx^2} - \frac{1}{8} \cdot \left(\frac{2Nx - P}{Mx^2}\right)^2 \&c.\right\} \\ = M^{\frac{1}{2}} \left\{x - \frac{N}{M} + \frac{P}{2M} \frac{1}{x} - \&c.\right\}; \end{aligned}$$

therefore by substitution in the original equation,

$$y = mx + n \pm \frac{M^{\frac{1}{2}}}{2a} \left\{ x - \frac{N}{M} + \frac{P}{2M} \cdot \frac{1}{x} - \&c. \right\}$$

Let x be now supposed to become infinitely great, then the terms on the right which have x in the denominator will vanish, and the ordinate to the hyperbola will coincide with that to the asymptotes; therefore the equation to the asymptotes is

$$y = mx + n \pm \frac{M^{\frac{1}{2}}}{2a} \left(x - \frac{N}{M} \right).$$

348. Cor. The asymptotes evidently intersect at the centre.

Because if x be supposed $= \frac{N}{M}$, the abscissa of the centre, then y becomes the corresponding ordinate of the diameter $y = mx + n$, that is, the ordinate of the centre.

Observation. Comparing the equation to the asymptotes with the equation to the two straight lines, which are one of the varieties of the hyperbola, we perceive that they are identical. See Art. 343.

349. If θ, θ' be the angles which the asymptotes make with the axis of x , then replacing m and M by their values,

$$\tan \theta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \tan \theta' = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

We may now determine the position of the asymptotes, corresponding to the different forms which the equation to the hyperbola may assume. See Art. 344.

1st. Let $a = 0$ or the equation be

$$bxy + cx^2 + dy + f = 0,$$

$$\text{then } \tan \theta = \frac{0}{0}, \quad \tan \theta' = -\frac{b}{0};$$

therefore the second asymptote is parallel to the axis of y .

To find the position of the first, we must put the expression for $\tan \theta$ under a different form. For this purpose, multiply numerator and denominator by

$$-b - \sqrt{b^2 - 4ac},$$

$$\text{then } \tan \theta = \frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})} = \frac{2c}{-b - \sqrt{b^2 - 4ac}};$$

therefore when $a = 0$

$$\tan \theta = -\frac{c}{b}.$$

2dly. Let $c = 0$ or the equation be

$$ay^2 + bxy + dy + ex + f = 0,$$

$$\text{then } \tan \theta = \frac{0}{2a}, \quad \tan \theta' = -\frac{b}{a};$$

therefore the first asymptote is parallel to the axis of x .

3dly. Let a and $c = 0$ at once, or the equation be

$$bxy + dy + ex + f = 0,$$

$$\text{then } \tan \theta = 0, \quad \tan \theta' = -\frac{b}{0},$$

whence the asymptotes are parallel to the axes of x and y .

350. From a comparison of this with the preceding Chapter it appears, that the sections of a cone are identical with the loci of the general equation of the second degree, between two variables: excluding from consideration two of these loci, namely 0 and two parallel straight lines, which, it is evident from the nature of the case, can never result from the intersection of a cone and a plane.

CHAPTER II.

ON THE DISCUSSION OF THE GENERAL EQUATION OF THE
SECOND DEGREE, BY THE TRANSFORMATION OF
CO-ORDINATES.

351. It was remarked in a former Article, that, whatever be the direction of the axes of co-ordinates, the Conic Sections may always be represented by one of the equations,

$$y^2 = qx, \text{ or } my^2 + nx^2 = p.$$

The principal object of this Chapter is to prove, that the general equation of the second degree between two variables is capable of being reduced, by the transformation of co-ordinates, to one of these simple forms.

It may not be superfluous to repeat, that the nature of a curve is not altered by transferring it from one system of axes to another. The purpose to which the transformation of co-ordinates is chiefly applied is to simplify the equation to any proposed curve, and consequently to facilitate the investigation of its several properties.

SECTION I.

ON THE REDUCTION OF THE GENERAL EQUATION.

352. *THE terms containing the simple powers of x and y in the general equation*

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

may be taken away, by changing the origin of the co-ordinates.

Let the curves which are the loci of the above equation be referred to any system of axes whatever; then the origin of the axes may be changed by means of the formulas, Art. 70,

$$x = x + \alpha, \quad y = y + \beta.$$

Substituting these values of x and y in the general equation, and arranging the result, we have

$$ay^2 + bxy + cx^2 + (2a\beta + b\alpha + d)y + (2c\alpha + b\beta + e)x + a\beta^2 + b\alpha\beta + c\alpha^2 + d\beta + e\alpha + f = 0.$$

Now in order that the terms involving the simple powers of x and y may = 0, we must have

$$2a\beta + b\alpha + d = 0,$$

$$\text{and } 2c\alpha + b\beta + e = 0.$$

From these equations

$$\alpha = \frac{2ae - bd}{b^2 - 4ac} \quad \text{and} \quad \beta = \frac{2cd - be}{b^2 - 4ac}.$$

These are the values of α, β , which render the proposed transformation possible; hence denoting the constant quantity

$$a\beta^2 + b\alpha\beta + c\alpha^2 + d\beta + e\alpha + f \text{ by } f',$$

the transformed equation becomes

$$ay^2 + bxy + cx^2 + f' = 0.$$

The general equation cannot therefore be reduced to this form, unless the preceding values of α and β remain finite.

When $b^2 - 4ac = 0$, the values of α and β become infinitely great, and the proposed transformation is impossible.

353. Cor. 1. Since the equation

$$ay^2 + bxy + cx^2 + f' = 0,$$

holds true if $-x$ and $-y$ be substituted for x and y , the new origin (α, β) bisects every chord drawn through it.

This may be directly proved.

Let any line $y = Ax$ be drawn through the origin: then the co-ordinates of its intersection with the curve will be obtained by combining $y = Ax$ with the equation

$$ay^2 + bxy + cx^2 + f' = 0.$$

We thus have

$$\{A^2a + Ab + c\}x^2 + f' = 0;$$

$$\therefore x = \pm \sqrt{\frac{-f'}{A^2a + Ab + c}} \quad \text{and} \quad y = \pm A \sqrt{\frac{-f'}{A^2a + Ab + c}}.$$

The abscissas and ordinates of the two points of intersection being equal, with contrary signs, it is evident that the chord is bisected in the point (a, β) .

The same being true of every other chord, the point (a, β) is *the centre*.

That this point is the centre might have being inferred at once from its co-ordinates a, β being identical with the co-ordinates of the centre, as obtained in the last Chapter.

354. COR. 2. Since the ellipse and hyperbola are characterized by $b^2 - 4ac$ being $<$ or > 0 respectively, it is evident that the general equation to these two curves, *when referred to their centre*, is

$$ay^2 + bxy + cx^2 + f = 0.$$

355. *The term containing xy in the general equation*

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

may be taken away, by passing from one system of rectangular axes to another.

If the curves which are the loci of the above equation be supposed to be referred to rectangular co-ordinates, they may be transferred to another rectangular system, originating at the same point, by means of the formulas, Art. 74,

$$x = x \cos \theta - y \sin \theta,$$

$$y = x \sin \theta + y \cos \theta.$$

Substituting these values of x and y in the general equation, and arranging the result, we have

$$\begin{aligned} & \{a \cos^2 \theta - b \sin \theta \cos \theta + c \sin^2 \theta\} y^2 \\ & + \{2a \sin \theta \cos \theta + b \cos^2 \theta - b \sin^2 \theta - 2c \sin \theta \cos \theta\} xy \\ & + \{a \sin^2 \theta + b \sin \theta \cos \theta + c \sin^2 \theta\} x^2 \\ & + (d \cos \theta - e \sin \theta) y + (d \sin \theta + e \cos \theta) x \\ & + f = 0. \end{aligned}$$

Now in order that the term involving xy may = 0, we must have

$$2a \sin \theta \cos \theta + b (\cos^2 \theta - \sin^2 \theta) - 2 \sin \theta \cos \theta = 0,$$

$$\text{or } (a - c) \sin 2\theta + b \cos 2\theta = 0,$$

$$\text{or } \tan 2\theta = -\frac{b}{a - c}.$$

But as the tangent of an angle may pass through all degrees of magnitude from 0 to ∞ , this value of $\tan 2\theta$ is always possible, whatever be the values of a , b and c .

Now since

$$\cos 2\theta = \frac{1}{\sec 2\theta} = \pm \sqrt{1 + \tan^2 2\theta},$$

we have

$$\cos 2\theta = \frac{a - c}{\pm \sqrt{(a - c)^2 + b^2}};$$

$\therefore \sin \theta$, which = $\sqrt{\frac{1}{2}(1 - \cos 2\theta)}$, and $\cos \theta$, which = $\sqrt{\frac{1}{2}(1 + \cos 2\theta)}$ may be determined.

The coefficients therefore of the transformed equation are known, and may be denoted by the single letters m , n , r , s ; the constant quantity f remaining the same.

Hence it appears that such a value of θ may always be found as shall render the above transformation possible, and reduce the general equation to the form

$$my^2 + nx^2 + ry + sx + f = 0 \dots \dots \dots (1).$$

356. Cor. 1. The new axes may be thus determined.

Conceive AX , AY to be the primitive axes; at the point A in the straight line AX make the angle XAZ equal to the angle whose tangent is $-\frac{b}{a - c}$, and bisect

the angle ZAX by the line AX' : then AX' will evidently be the new axis of x , and if AY' be drawn at right angles to AX' , it will be the new axis of y .

357. COR. 2. Since

$$\sin 2\theta = \cos 2\theta \tan 2\theta,$$

we have by substitution,

$$\sin 2\theta = \frac{-b}{\pm \sqrt{(a-c)^2 + b^2}}.$$

In order that $\sin 2\theta$ may always be positive, the numerator and denominator must have the same sign: in other words, we must take the positive or the negative sign of the radical quantity, according as b in the original equation is negative or positive.

358. *To find the values of the coefficients m and n of the transformed equation, in terms of a , b and c .*

$$\text{Since } m = a \cos^2 \theta - b \sin \theta \cos \theta + c \sin^2 \theta,$$

$$\text{and } n = a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta,$$

we have by addition, observing that $\sin^2 \theta + \cos^2 \theta = 1$,

$$m + n = a + c, \dots \dots \dots (1);$$

and by subtraction, observing that $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$,

$$m - n = (a - c) \cos 2\theta - b \sin 2\theta;$$

therefore substituting for $\cos 2\theta$ and $\sin 2\theta$ their values in Arts. 355 and 357, we have

$$m - n = \pm \sqrt{(a - c)^2 + b^2} \dots \dots \dots (2);$$

hence adding and subtracting successively (1) and (2),

$$m = \frac{1}{2} \{a + c \pm \sqrt{(a - c)^2 + b^2}\},$$

$$\text{and } n = \frac{1}{2} \{a - c \mp \sqrt{(a - c)^2 + b^2}\}.$$

359. **COR.** Multiplying together these values of m and n , we have

$$mn = \frac{1}{4} (4ac - b^2),$$

$$\text{or } b^2 - 4ac = -4mn.$$

Observation. It may be shewn in like manner, but the process is very operose, that if the original axes be oblique, and the curve be transferred to another oblique system, by means of the formulas, Art. 71,

$$x = \frac{1}{\sin \gamma} \{x \sin (\gamma - \theta) + y \sin (\gamma - \theta')\},$$

$$y = \frac{1}{\sin \gamma} \{x \sin \theta + y \sin \theta'\};$$

the second term may be taken away, and the curve reduced to the form

$$m'y^2 + n'x^2 + r'y + s'x + f = 0.$$

We may assume therefore that, whatever be the inclination of the axes, the general equation may be reduced to the form

$$my^2 + nx^2 + ry + sx + f = 0.$$

360. *When the curve is represented by the equation*

$$my^2 + nx^2 + ry + sx + f = 0,$$

the axes are parallel to conjugate diameters.

For let $y = \beta'$ be any chord parallel to the axis of x , then the abscissas of its intersection with the curve will be obtained from the equation

$$nx^2 + sx + m\beta'^2 + r\beta' + f = 0;$$

therefore the abscissa of its middle point $= -\frac{s}{2n}$, which being constant, the diameter which bisects chords parallel to the axis of x is parallel to the axis of y .

Similarly, the diameter which bisects chords parallel to the axis of y is parallel to the axis of x .

These diameters are therefore conjugate to each other : (See Arts. 166 and 259), but the axes are parallel to them ; therefore the equation

$$my^2 + nx^2 + ry + sx + f = 0,$$

characterizes the curve, when it is referred to axes parallel to a system of conjugate diameters.

SECTION II.

ON THE REDUCTION OF THE EQUATION TO THE PARABOLA.

361. It was shewn that, when the axes are parallel to a system of conjugate diameters, the general equation is always reducible to the form

$$my^2 + nx^2 + ry + sx + f = 0,$$

in which

$$m = \frac{1}{2} \{a + c \pm \sqrt{(a - c)^2 + b^2}\}$$

$$n = \frac{1}{2} \{a + c \mp \sqrt{(a - c)^2 + b^2}\}.$$

Now in the parabola $b^2 = 4ac$;

$$\therefore (a - c)^2 + b^2 = (a + c)^2;$$

$$\therefore m = a + c \text{ or } 0,$$

$$\text{and } n = 0 \text{ or } a + c;$$

therefore in the case of the parabola, when the term involving xy is removed, the resulting equation is of the form

$$my^2 + ry + sx + f = 0,$$

$$\text{or } nx^2 + ry + sx + f = 0.$$

362. Cor. 1. It appears from a former Article that when

$$m = a + c \text{ and } n = 0,$$

b must be negative,

and that when $n = a + c$ and $m = 0$,

b must be positive.

363. COR. 2. It is evident that m and n cannot $= 0$ at the same time.

364. COR. 3. To find the values of r and s .

$$r = d \cos \theta - e \sin \theta, \text{ and } s = d \sin \theta + e \cos \theta.$$

Now

$$\cos 2\theta = \frac{a - c}{\pm \sqrt{(a - c)^2 + b^2}} = \frac{a - c}{\pm (a + c)}, \text{ since } b^2 = 4ac;$$

therefore taking the positive value $a + c$, which corresponds to b negative,

$$\cos \theta = \sqrt{\frac{1}{2} (1 + \cos 2\theta)} = \sqrt{\frac{a}{a + c}},$$

$$\text{and } \sin \theta = \sqrt{\frac{1}{2} (1 - \cos 2\theta)} = \sqrt{\frac{c}{a + c}};$$

$$\therefore r = \frac{d \sqrt{a - e \sqrt{c}}}{\sqrt{a + c}}, \text{ and } s = \frac{d \sqrt{c + e \sqrt{a}}}{\sqrt{a + c}}.$$

In like manner, if the negative value $-(a + c)$ be taken, which corresponds to b positive,

$$r = \frac{d \sqrt{c - e \sqrt{a}}}{\sqrt{a + c}}, \text{ } s = \frac{d \sqrt{a + e \sqrt{c}}}{\sqrt{a + c}}.$$

365. *The equation to the parabola may always be reduced to the form*

$$y^2 = qx, \text{ or } x^2 = q'y.$$

First let the equation be

$$my^2 + ry + sx + f = 0;$$

therefore substituting $y + \beta$ and $x + \alpha$ for y and x , we have

$$m(y + \beta)^2 + r(y + \beta) + s(x + \alpha) + f = 0,$$

$$\text{or } my^2 + (2m\beta + r)y + sx + r\beta^2 + s\alpha + f = 0,$$

in which it is manifest that the term involving x cannot $= 0$, unless $s = 0$, which is contrary to the hypothesis.

In order that the term involving y , and the terms which are independent of x and y , may $= 0$, we must have

$$2m\beta + r = 0, \text{ and } r\beta^2 + s\alpha + f = 0;$$

$$\therefore \beta = -\frac{r}{2m} \text{ and } \alpha = -\frac{r\beta^2 + f}{s},$$

both which values are real and finite.

The proposed equation becomes therefore

$$my^2 + sx = 0, \text{ or } y^2 = -\frac{s}{m}x,$$

which is of the form

$$y^2 = qx.$$

In like manner it may be shewn, that the second of the equations in Art. 361 may be reduced to the form

$$x^2 = q'y.$$

366. COR. 1. These two equations represent the same curve.

If the former $y^2 = qx$ be the equation to the curve, when the axes are AX, AY , the latter $x^2 = q'y$ will be its equation when the position of the axes is reversed.

367. COR. 2. Since the equation $y^2 = qx$ has no constant term, if we suppose

$$y \text{ or } x = 0, \text{ then } x \text{ or } y = 0;$$

therefore the origin is upon the curve.

Also, the new axes being by hypothesis parallel to the old ones, the new axis of x is a diameter, and the new axis of y becomes a tangent to the curve.

Hence, when the parabola is referred to any diameter, and the tangent at its extremity, as axes, its equation is of the form

$$y^2 = qx.$$

368. COR. 3. If r be supposed $= 0$, then β would $= 0$, and α would $= -\frac{f}{s}$: therefore the new origin would in this case be situated on the axis of x , its position relatively to the old origin depending upon the sign of α .

369. COR. 4. We cannot assume that m and r , or that n and s are both $= 0$ at the same time: for then

$$nx^2 + sx + f \text{ would } = 0, \text{ or } x = -\frac{s}{2n} \pm \frac{1}{2n} \sqrt{s^2 - 4nf},$$

and

$$my^2 + ry + f \text{ would } = 0, \text{ or } y = -\frac{r}{2m} \pm \frac{1}{2m} \sqrt{r^2 - 4mf}.$$

The former is the equation to two straight lines parallel to the axis of y , the latter to two straight lines parallel to the axis of x .

Hence, in the case supposed, the equation would no longer represent a line of the second order, but one of its varieties. See *Observation*, Art. 337.

370. It appears therefore that the equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

when it represents a parabola, may be successively reduced, according as b is in either case negative or positive,

(1) to the form

$$my^2 + ry + sx + f = 0,$$

or

$$nx^2 + ry + sx + f = 0,$$

when the axes are parallel to conjugate diameters.

(2) to the form

$$my^2 + sx = 0, \quad \text{or} \quad y^2 = qx,$$

or

$$nx^2 + ry = 0, \quad \text{or} \quad x^2 = q'y,$$

when the axes are any diameter and the tangent at its extremity.

SECTION III.

ON THE REDUCTION OF THE EQUATION TO THE ELLIPSE AND HYPERBOLA.

371. It was shewn that the general equation is always reducible to the form

$$my^2 + nx^2 + ry + sx + f = 0,$$

and since $b^2 - 4ac$, which $= -4mn$, must be $<$ or > 0 , it is evident that neither m nor n can $= 0$, in the above equation, when it represents the ellipse and the hyperbola.

372. *When the equation to the ellipse or hyperbola wants the term involving xy , it may always be reduced to the form*

$$my^2 + nx^2 = p,$$

by changing the origin of the axes.

The direction of the axes remaining the same, let their origin be transferred to a new point (α, β) , by means of the formulas

$$x = x + \alpha \text{ and } y = y + \beta.$$

Substituting these values for x and y in the equation

$$my^2 + nx^2 + ry + sx + f = 0,$$

expanding, and arranging the result, we have

$$\begin{aligned} my^2 + nx^2 + (2m\beta + r)y + (2n\alpha + s)x \\ + m\beta^2 + n\alpha^2 + r\beta + s\alpha + f = 0. \end{aligned}$$

Now in order that the terms involving x and y may = 0, we must have

$$2m\beta + r = 0, \text{ and } 2n\alpha + s = 0;$$

$$\therefore \beta = -\frac{r}{2m}, \text{ and } \alpha = -\frac{s}{2n},$$

which values being real and finite, the transformation is always possible.

Such values therefore may be assigned to α , β , the co-ordinates of the new origin, as shall reduce the equation to the ellipse and hyperbola to the form

$$my^2 + nx^2 = p,$$

$$p \text{ being assumed} = -\{m\beta^2 + n\alpha^2 + r\beta + s\alpha + f\}.$$

373. COR. The new axes are evidently a system of conjugate diameters.

374. *When the equation to the ellipse or hyperbola wants the terms involving the simple powers of x and y , it may always be reduced to the form*

$$my^2 + nx^2 = p',$$

by passing from one rectangular system to another.

The origin of the axes remaining the same, their direction may be changed from one rectangular system to another, by means of the formulas

$$x = x \cos \theta - y \sin \theta,$$

$$y = x \sin \theta + y \cos \theta.$$

Substituting these values of x and y in the proposed equation

$$ay^2 + bxy + cx^2 + f = 0,$$

and reducing it as in Section 1, where the operation is performed for the equation when it has all its terms, we find

$$\tan 2\theta = -\frac{b}{a-c};$$

and the transformed equation becomes

$$my^2 + nx^2 = p',$$

p' being assumed $= -f'$.

375. *Cor.* It is evident that, as in the preceding Proposition, the new axes are a system of conjugate diameters.

376. It appears therefore that the equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

when it represents an ellipse or hyperbola, may be successively reduced :

(1) to the form

$$ay^2 + bxy + cx^2 + f' = 0,$$

when the curve is referred to the centre, and any system of axes whatever.

And

(2) to the form

$$my^2 + nx^2 = p',$$

when the curve is referred to the centre, and any system of conjugate diameters.

377. *Having given the equation to the ellipse and hyperbola, when referred to the centre and any system of conjugate diameters, to find their equation, when they are referred to the centre and the principal axes.*

We shall first take the case of the ellipse.

If $2a'$, $2b'$ be any system of conjugate diameters, its equation will be

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2 \dots\dots\dots (1).$$

To transfer this to a rectangular system, we must employ the formulas

$$y = \frac{1}{\sin \gamma} \{x \sin \theta + y \cos \theta\};$$

$$x = \frac{1}{\sin \gamma} \{x \sin (\gamma - \theta) - y \cos (\gamma - \theta)\};$$

therefore substituting these values of x and y in (1), we have

$$\begin{aligned} & \frac{a'^2}{\sin^2 \gamma} \{x^2 \sin^2 \theta + 2xy \sin \theta \cos \theta + y^2 \cos^2 \theta\} \\ & + \frac{b'^2}{\sin^2 \gamma} \{x^2 \sin^2 (\gamma - \theta) - 2xy \sin (\gamma - \theta) \cos (\gamma - \theta) + y^2 \cos^2 (\gamma - \theta)\} \\ & = a'^2 b'^2; \end{aligned}$$

$$\begin{aligned} \therefore \{a'^2 \cos^2 \theta + b'^2 \cos^2 (\gamma - \theta)\} y^2 & + \{a'^2 \sin^2 \theta + b'^2 \sin^2 (\gamma - \theta)\} x^2 \\ & + 2 \{a'^2 \sin \theta \cos \theta - b'^2 \sin (\gamma - \theta) \cos (\gamma - \theta)\} xy = a'^2 b'^2 \sin^2 \gamma. \end{aligned}$$

Now since the new axes are by hypothesis conjugate diameters, the term containing $xy = 0$;

$$\therefore a'^2 \sin \theta \cos \theta - b'^2 \sin (\gamma - \theta) \cos (\gamma - \theta) = 0 \dots\dots (1);$$

therefore the transformed equation is

$$\begin{aligned} \{a'^2 \cos^2 \theta + b'^2 \cos^2 (\gamma - \theta)\} y^2 & + \{a'^2 \sin^2 \theta + b'^2 \sin^2 (\gamma - \theta)\} x^2 \\ & = a'^2 b'^2 \sin^2 \gamma \dots\dots\dots (2). \end{aligned}$$

Here it only remains to determine θ , which is the angle contained between the new axis of x and the old one.

Substituting in equation (1)

$$\begin{aligned} \frac{1}{2} \sin 2\theta \text{ for } \sin \theta \cos \theta, \text{ and } \frac{1}{2} \sin (2\gamma - 2\theta) \text{ for} \\ \sin (\gamma - \theta) \cos (\gamma - \theta), \end{aligned}$$

we have

$$a'^2 \sin 2\theta - b'^2 \sin (2\gamma - 2\theta) = 0;$$

$$\text{or } a'^2 \sin 2\theta - b'^2 \{\sin 2\gamma \cos 2\theta - \cos 2\gamma \sin 2\theta\} = 0;$$

therefore dividing by $\cos 2\theta$,

$$a'^2 \tan 2\theta - b'^2 \sin 2\gamma + b'^2 \cos 2\gamma \tan 2\theta = 0;$$

$$\therefore \tan 2\theta = \frac{b'^2 \sin 2\gamma}{a'^2 + b'^2 \cos 2\gamma},$$

therefore 2θ is known.

The straight line bisecting the angle 2θ is one of the principal axes, and the other is at right angles to it.

In a similar manner the equation may be resolved for the hyperbola, whose equation is

$$a'^2 y^2 - b'^2 x^2 = -a'^2 b'^2.$$

378. COR. As there is but one system of rectangular conjugate diameters, namely the axes, the transformed equation must be absolutely identical with the equation

$$a^2 y^2 + b^2 x^2 = a^2 b^2;$$

therefore equating the corresponding terms, we have

$$a'^2 \cos^2 \theta + b'^2 \cos^2 (\gamma - \theta) = a^2 \dots \dots \dots (1),$$

$$a'^2 \sin^2 \theta + b'^2 \sin^2 (\gamma - \theta) = b^2 \dots \dots \dots (2),$$

$$a'^2 b'^2 \sin^2 \gamma = a^2 b^2 \dots \dots \dots (3).$$

Adding the first two we have, observing that $\sin^2 \theta + \cos^2 \theta = 1$,

$$\left. \begin{aligned} a'^2 + b'^2 &= a^2 + b^2 \\ a'b' \sin \gamma &= ab \end{aligned} \right\} \text{as in Arts. 185 and 186.}$$

In a precisely similar manner it may be shewn, that in the hyperbola

$$\left. \begin{aligned} a'^2 - b'^2 &= a^2 - b^2 \\ a'b' \sin \gamma &= ab \end{aligned} \right\} \text{as in Arts. 278 and 280.}$$

CHAPTER III.

ON THE APPLICATION OF THE GENERAL EQUATION OF THE SECOND DEGREE.

379. *A STRAIGHT line cannot cut a line of the second order in more than two points.*

The co-ordinates of the points in which a straight line

$$y = ax + \beta,$$

cuts a line of the second order

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

will be determined by combining these two equations.

Substituting therefore in the latter the value of y derived from the former, we have

$$\{aa^2 + ba + c\}x^2 + \{(2aa + b)\beta + da + e\}x + a\beta^2 + d\beta + f = 0.$$

This equation being of the second degree can have only two roots, which, when real, represent the abscissas of the points of intersection. Hence the truth of the Proposition.

380. *To find the equation to the tangent at a point (x', y') in a line of the second order.*

The curve being referred to any axes whatever, let (x'', y'') be another point near the first: then the equation to the straight line drawn through

$$(x', y') \text{ and } (x'', y'')$$

and cutting the curve, is

$$y - y' = \frac{y' - y''}{x' - x''} (x - x') \dots \dots \dots (1).$$

Now the two points (x', y') and (x'', y'') being in the curve, we must have

$$ay'^2 + bx'y' + cx'^2 + dy' + ex' + f = 0,$$

$$\text{and } ay''^2 + bx''y'' + cx''^2 + dy'' + ex'' + f = 0;$$

$$\therefore a(y'^2 - y''^2) + b(x'y' - x''y'') + c(x'^2 - x''^2) \\ + d(y' - y'') + e(x' - x'') = 0 \dots \dots \dots (2).$$

$$\text{But } y'^2 - y''^2 = (y' + y'')(y' - y''),$$

$$x'y' - x''y'' = x'(y' - y'') + y''(x' - x''),$$

$$x'^2 - x''^2 = (x' + x'')(x' - x'');$$

therefore substituting these values in (2), and collecting the terms which involve $y' - y''$ and $x' - x''$, we have

$$(y' - y'')\{a(y' + y'') + bx' + d\} + (x' - x'')\{by'' + c(x' + x'') + e\} = 0,$$

$$\therefore \frac{y' - y''}{x' - x''} = - \frac{by'' + c(x' + x'') + e}{a(y' + y'') + bx' + d}.$$

This value of $\frac{y' - y''}{x' - x''}$ being substituted in (1) the equation to the line cutting the curve in the points (x', y') and (x'', y'') is found.

Suppose now that the points of section coincide, then $x' = x''$, $y' = y''$ and the cutting line becomes a tangent:

therefore the equation to the tangent is

$$y - y' = -\frac{by' + 2cx' + e}{2ay' + bx' + d}(x - x');$$

or since

$$ay'^2 + bx'y' + cx'^2 + dy' + ex' + f = 0,$$

the equation to the tangent becomes

$$(2ay' + bx' + d)y + (2cx' + by' + e)x + dy' + ex' + 2f = 0.$$

381. *To find when a line of the second order intersects the axes of co-ordinates.*

The general equation being

$$ax^2 + bxy + cy^2 + dx + ey + f = 0;$$

let the locus be supposed to cut the axis of x , then $y = 0$;

$$\therefore cx^2 + ex + f = 0;$$

$$\therefore x = -\frac{e}{2c} \pm \frac{1}{2c}\sqrt{e^2 - 4cf},$$

which are the abscissas of the points of intersection.

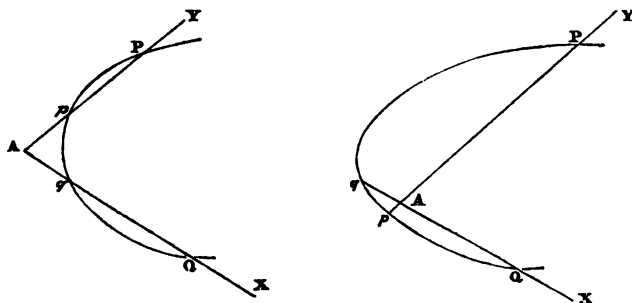
In order therefore that the intersection may take place, e^2 must $> 4cf$.

If $e^2 < 4cf$, the values of x are imaginary, and the curve does not meet the axis of x .

If $e^2 = 4cf$, the two values of x are equal, hence the points of intersection coincide, and the locus *touches* the axis of x .

In like manner it may be proved, that the locus intersects, falls without, or touches, the axis of y , according as d^2 is $>$ or $<$ or $= 4af$.

382. *If through any point within or without a line of the second order two straight lines be drawn cutting the curve, the rectangle contained by the segments of the one will bear a constant ratio to the rectangle contained by the segments of the other.*



Let A be any point within or without a line of the second order, and through A let AP , AQ be drawn cutting the curve in the points P , p and Q , q : then

$AP \cdot Ap$ will be to $AQ \cdot Aq$ in a constant ratio.

Whatever be the inclination of AP and AQ to each other, if the curve be referred to these lines as axes, its equation will be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

In this equation,

Let $x = 0$, then the curve cuts the axis of y ;

$$\therefore ay^2 + dy + f = 0,$$

in which the two values of y are AP , Ap ; therefore by the theory of equations

$$AP \cdot Ap = \frac{f}{a} \dots \dots \dots (1).$$

Next let $y = 0$, then the curve cuts the axis of x ;

$$\therefore cx^2 + ex + f = 0,$$

in which the two values of x are AQ , Aq ;

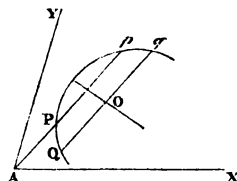
$$\therefore \text{ as before, } AQ \cdot Aq = \frac{f}{c} \dots \dots \dots (2),$$

$$\therefore AP \cdot Ap : AQ \cdot Aq :: \frac{f}{a} : \frac{f}{c} :: c : a.$$

Now whatever be the position of A , the values of a and c in the general equation remain constant, so long as the axes retain the same given inclination, Ch. II. Sect. I. therefore the rectangles $AP \cdot Ap$ and $AQ \cdot Aq$ are to each other in a constant ratio, as was to be proved.

Observation. This Proposition has already been proved in the particular cases of the circle, the parabola, the ellipse and the hyperbola. See Arts. 53, 120, 180 and 275.

383. *Any number of parallel chords being drawn in a line of the second order, to find the locus of their middle points.*



Let QPp be any line of the second order referred to any axes whatever AX, AY .

Through A draw any line APp , meeting the curve in the points P, p , and draw any chord Qq parallel to Pp : bisect Qq in O , then the question is to find the equation to the point O .

Let the co-ordinates of O be x', y' , and suppose the equation to APp to be $y = mx$.

If the origin be transferred to O , the general equation becomes

$$a(y + y')^2 + b(x + x')(y + y') + c(x + x')^2 + d(y + y') + e(x + x') + f = 0 \dots \dots \dots (1),$$

and the equation to Qq , drawn through the origin and parallel to APp , will be

$$y = mx \dots \dots \dots (2)$$

When Qq intersects the curve, the co-ordinates in these two equations will be identical; therefore the values of the abscissa at the points of intersection will be determined from the equation

$$\begin{aligned} a(mx + y')^2 + b(x + x')(mx + y') + c(x + x') + d(mx + y') \\ + e(x + x') + f = 0. \end{aligned}$$

But Qq being bisected in O , the roots of this equation will be equal, with opposite signs; therefore the coefficient of its second term, that is of x , must = 0.

Hence

$$2am y' + b y' + b m x' + 2c x' + d m + e = 0;$$

therefore arranging, and suppressing the accents, we have

$$(2am + b)y + (bm + 2c)x + dm + e = 0,$$

which is the equation to a straight line.

The locus, therefore, of the middle points of all parallel chords in a line of the second order is a straight line.

The straight line which has thus been shewn to bisect any number of parallel chords is called a *diameter*: and the points in which it meets the curve are called the *vertices*.

384. COR. 1. Let the diameter be parallel to AX , then $m = 0$,

$$\therefore by + 2cx + e = 0 \dots \dots \dots (1),$$

is the equation to the diameter which is parallel to the axis of x .

Next suppose the diameter parallel to AY , then

$$m \text{ which } = \frac{\sin \rho, x}{\sin \rho, y} \text{ will } = \frac{\sin \rho, x}{0},$$

which is infinitely great.

COR. 2. If $y = m'x$ be any other chord, the diameter which bisects it will be

$$(2am' + b)y + (bm' + 2c)x + dm' + e = 0;$$

therefore the finite quantities b , $2c$ and e vanishing when compared with m , the equation becomes .

$$2amy + bmx + dm = 0,$$

$$\text{or } 2ay + bx + d = 0 \dots \dots \dots (2),$$

which is the equation to the diameter parallel to the axis of y .

385. *If either of the diameters*

$$(2am + b)y + (bm + 2c)x + dm + e = 0$$

$$(2am' + b)y + (bm' + 2c)x + dm' + e = 0,$$

be parallel to the corresponding chords of the other, then reciprocally the other will be parallel to the corresponding chords of the former.

Let the first of these, for example, be parallel to the chord of the second, then

$$-\frac{bm + 2c}{2am + b} = m', \text{ Art. 25;}$$

$$\therefore m = -\frac{bm' + 2c}{2am' + b};$$

whence the second is parallel to the chord of the first; as was to be proved.

Two diameters so related, that each bisects the chords which are parallel to the other, are called *conjugate diameters*.

386. COR. The equation to any diameter being

$$y = mx + n;$$

the equation to the diameter conjugate to it will be

$$y = -\frac{bm + 2c}{2am + b}x + n'.$$

Whence it is manifest that the number of pairs of conjugate diameters is unlimited.

387. To find whether any system of conjugate diameters can be rectangular.

If $y = mx + n$, and $y = m'x + n'$ be any two conjugate diameters,

$$\text{then } m' = -\frac{bm + 2c}{2am + b};$$

$$\therefore mm' + \frac{b}{2a}(m + m') + \frac{c}{a} = 0 \dots \dots \dots (1).$$

Now in order that the proposed diameters may be at right angles to each other,

$$m' \text{ must} = -\frac{1 + m \cos \gamma}{m + \cos \gamma}; \quad \text{Art. 34.}$$

$$\therefore mm' + (m + m') \cos \gamma + 1 = 0 \dots \dots \dots (2),$$

whence m and m' may be found from equations (1) and (2).

It is evident from the symmetrical form of these, that the quadratic equations which serve to determine m and m' will be identical: hence since m and m' are the roots of the same quadratic equation, there can be but one system of conjugate diameters which are rectangular.

These are called the *principal diameters*, or *the axes*, and the points in which they meet the curve are called the *principal vertices*.

388. *To prove, independently of the transformation of co-ordinates, that the general equation may be so reduced as to coincide with the equation.*

$$y^2 = mx + nx^2.$$

It has been already shewn, that

$$2cx + by + e = 0 \dots \dots \dots (1),$$

is the diameter which bisects chords parallel to the axis of x , and that

$$2ay + bx + d = 0 \dots \dots \dots (2),$$

is the diameter which bisects chords parallel to the axis of y .

Suppose these diameters to be now conjugate to each other ;

Then the first being parallel to the axis of y , x must be equal to a constant quantity in (1) ;

$$\therefore b = 0.$$

Also the second being parallel to the axis of x , y must be equal to a constant quantity in (2) ;

$$\therefore b = 0.$$

Hence, *when the axes are parallel to conjugate diameters*, the equation is reduced to the form

$$ay^2 + cx^2 + dy + ex + f = 0.$$

Next, let the second diameter which is parallel to the axis of x coincide with it, then $d = 0$, and the equation becomes

$$ay^2 + cx^2 + ex + f = 0.$$

Lastly, if the origin be now supposed to be on the curve, $f = 0$; and in this case the axis of y will become a tangent.

Hence, *when the curve is referred to any diameter and the tangent at its extremity as axes*, the general equation is of the form

$$ay^2 + cx^2 + ex = 0,$$

which is identical with the equation

$$y^2 = mx + nx^2.$$

Cor. Let the centre be the origin.

In this case equations (1) and (2) must be of the form

$$y = Mx;$$

therefore e in (1) and d in (2) must = 0;

therefore the equation is reduced to the form

$$ay^2 + cx^2 + f = 0.$$

THE END.

